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A TRACT

ON THE ADDITION OF

ELLIPTIC AND HYPER-ELLIPTIC INTEGRALS.

BY

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PREFACE.

THIS little work was written with the view of presenting in a simple and elementary way the fundamental properties of those transcendents which, in the ascending order of classification, immediately succeed elliptic functions. I have adopted as the basis of my investigations the following memoirs of Jacobi, "Considerationes generales de transcendentibus Abelianis" (*Crelle*, vol. 9, p. 394), "De functionibus quadrupliciter periodicis, quibus theoria transcendentium Abelianarum innititur" (*Crelle*, vol. 13, p. 55), "Demonstratio nova theorematis Abeliani" (*Crelle*, vol. 24, p. 28), and of the greater part of these memoirs much of what appears in the following pages may be regarded as a free translation. By the employment of the method given in the last of the above-mentioned memoirs, I have constructed a trigonometry of the functions in question, by which the reader is introduced to the study of them in the same way as he has already become acquainted with the properties of circular functions by the formulæ of ordinary trigonometry. Since the appearance of Jacobi's papers, the subject has received large developments at the hands of Riemann, Weierstrass, and other distinguished mathematicians, and I venture to hope that what I have put together may invite the attention of mathematical students to the higher theories and more abstract principles of the German analysts.

The perfect uniformity of treatment of elliptic and hyper-elliptic functions, by which their fundamental properties are derived from the employment of Jacobi's method, is shewn in the second, third, fourth, and fifth chapters; the sixth chapter is devoted to an accurate examination of those cases where the first class of hyper-elliptic integrals depends on elliptic functions, a question which appears, ever since the subject was started, to have been invested with interest, and the analogues of Fagnano's theorem, which I have given in the seventh chapter, bear out the well-known analogy between conic sections and the lines of curvature of an ellipsoid.

I have to acknowledge, in the first place, my obligations to Mr. Cathcart, Fellow of Trinity College, Dublin, for the readiness with which he undertook the revision of the proof sheets, by which greater freedom from typographical errors has been attained than could have been secured without such aid, and also for suggestions of which I gladly availed myself.

I have also to thank the Board of Trinity College for their liberality in contributing to the expense of publication.

TRINITY COLLEGE, DUBLIN,
June 21, 1871.

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TRACT ON THE ADDITION OF THE ELLIPTIC AND HYPERELLIPTIC INTEGRALS.

ERRATA.

Page 72, line 2, *for* $P\left(-\kappa''^2, \frac{\sqrt{(\kappa''^2-\kappa^2)}}{k'}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \theta'\right)$,

read $P\left(-\kappa''^2, \frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \theta'\right)$.

Page 82, line 3, *for* Mathematica, *read* Matematica.

" " line 9, *for* $(\alpha'+\beta'y)\,\partial y$, *read* $(\alpha'+\beta'y)\,dy$.

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property of the function of x whose differential coefficie
$\frac{1}{x}$ from the addition of two such functions. Let $f(x)=\int_1^x$
then the differential equation $\frac{dx_1}{x_1}+\frac{dx_2}{x_2}=0$ has a transce
dental integral $f(x_1)+f(x_2)=c$, as well as the algebraic integral $x_1x_2=c'$. The constants in either form are determined by the supposition that we are acquainted with simultaneous values of x_1 and x_2. If when $x_1=1$, x_2 becomes α, we have $f(x_1)+f(x_2)=f(\alpha)$, $x_1x_2=\alpha$, and on these equations the whole theory of logarithms can be based.

Again, let $F(x)=\int_0^x\frac{dx}{1+x^2}$, and the addition of two functions of the form $F(x)$ depends on the determination under

TRACT ON THE ADDITION OF THE ELLIPTIC AND HYPERELLIPTIC INTEGRALS.

CHAPTER I.

JACOBI'S PROOF OF ABEL'S THEOREM.

1. THE functions which I intend to consider in this tract are comprised under a remarkable class of transcendents, namely, those whose differential coefficients are algebraic. The solution of the problem of their addition depends in general on the determination under an algebraic form of the integrals of a system of differential equations of the first order. Suppose, for instance, that we had no previous knowledge on the subject, we could derive the fundamental property of the function of x whose differential coefficient is $\frac{1}{x}$ from the addition of two such functions. Let $f(x) = \int_1^x \frac{dx}{x}$: then the differential equation $\frac{dx_1}{x_1} + \frac{dx_2}{x_2} = 0$ has a transcendental integral $f(x_1) + f(x_2) = c$, as well as the algebraic integral $x_1 x_2 = c'$. The constants in either form are determined by the supposition that we are acquainted with simultaneous values of x_1 and x_2. If when $x_1 = 1$, x_2 becomes a, we have $f(x_1) + f(x_2) = f(a)$, $x_1 x_2 = a$, and on these equations the whole theory of logarithms can be based.

Again, let $F(x) = \int_0^x \frac{dx}{1+x^2}$, and the addition of two functions of the form $F(x)$ depends on the determination under an algebraic form of the integral of the differential equation $\frac{dx_1}{1+x_1^2}+\frac{dx_2}{1+x_2^2}=0$, which is easily seen to be $\frac{x_1+x_2}{1-x_1x_2}=c$, and if α is the value of x_2 when $x_1=0$, the transcendental equation $F(x_1)+F(x_2)=F(\alpha)$ connects x_1 and x_2 by the algebraic relation $\frac{x_1+x_2}{1-x_1x_2}=\alpha$, and the whole theory of circular functions can be derived from this property. After this illustration of what is meant by the addition of transcendental functions, I proceed to the consideration of Jacobi's proof of Abel's theorem, which contains the solution of the problem of the addition of a most important class of transcendents.

2. The analysis employed in this investigation involves some properties of rational fractions, which, in the first place, I proceed to demonstrate.

Let $$(x-x_1)(x-x_2)\ldots(x-x_m)=\phi(x),$$

let also $\psi(x)$ denote any rational and integral function of x; then, by the known theory of the decomposition of rational fractions,

$$\frac{\psi(x)}{\phi(x)}=A+\Sigma\,\frac{\psi(x_1)}{\phi'(x_1)}\cdot\frac{1}{x-x_1}\;\ldots\ldots\ldots\ldots\;(1),$$

where A is a rational and integral function of x, if the degree of $\psi(x)$ exceeds m, or the number of the quantities x_1, x_2; $\phi'(x)$ denotes, as usual, the first derived function of $\phi(x)$, and the sign of summation Σ extends to the quantities $x_1, x_2\ldots x_m$.

Now $$\frac{d^{m-1}}{dx_2dx_3\ldots dx_m}\left\{\frac{1}{\phi'(x_1)}\right\}=\left\{\frac{1}{\phi'(x_1)}\right\}^2,$$

also $$\frac{d^m}{dx_1dx_2\ldots dx_m}\left\{\frac{\psi(x)}{\phi(x)}\right\}=\frac{\psi(x)}{\{\phi(x)\}^2},$$

so that operating with $\frac{d^m}{dx_1dx_2\ldots dx_m}$ on (1) we find

$$\frac{\psi(x)}{\{\phi(x)\}^2}=\frac{d^m(A)}{dx_1dx_2\ldots dx_m}+\Sigma\,\frac{d}{dx_1}\left[\frac{1}{(x-x_1)}\,\frac{\psi(x_1)}{\{\phi'(x_1)\}^2}\right],$$

whence
$$\frac{\psi(x)}{\{\phi(x)\}^2} = \frac{d^m A}{dx_1 dx_2 \ldots dx_m} + \Sigma \frac{1}{(x-x_1)^2} \frac{\psi(x_1)}{\{\phi'(x_1)\}^2}$$
$$+ \Sigma \frac{1}{(x-x_1)} \frac{d}{dx_1} \frac{\psi(x_1)}{\{\phi'(x_1)\}^2}.$$

It follows from this last formula, that if $\frac{\psi(x)}{\{\phi(x)\}^2}$ be developed by descending powers of x, the coefficient of $\frac{1}{x}$ is
$$\Sigma \frac{d}{dx_1} \left[\frac{\psi(x_1)}{\{\phi'(x_1)\}^2}\right].$$

Now let $\psi(x) = A_0 + A_1 x + A_2 x^2 + \ldots + A_{2m-2} x^{2m-2}$,

then
$$\frac{\psi(x)}{\{\phi(x)\}^2} = \frac{\frac{A_0}{x^{2m}} + \frac{A_1}{x^{2m-1}} + \ldots + \frac{A_{2m-2}}{x^2}}{\left\{\left(1-\frac{x_1}{x}\right)\left(1-\frac{x_2}{x}\right)\ldots\left(1-\frac{x_m}{x}\right)\right\}^2},$$

and the right-hand side of this equation being developed by descending powers of x, the coefficient of $\frac{1}{x}$ does not appear: so that when $\psi(x)$ is of the degree $2m-2$ in x,
$$\Sigma \frac{d}{dx_1} \frac{\psi(x_1)}{\{\phi'(x_1)\}^2} = 0.$$

3. I now propose to lay before the reader the manner in which Jacobi has determined the algebraic integrals of the following system of equations:
$$\left.\begin{array}{l}
\frac{dx_1}{\sqrt{\{f(x_1)\}}} + \frac{dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \frac{dx_m}{\sqrt{\{f(x_m)\}}} = 0 \\
\frac{x_1 dx_1}{\sqrt{\{f(x_1)\}}} + \frac{x_2 dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \frac{x_m dx_m}{\sqrt{\{f(x_m)\}}} = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{x_1^{m-2} dx_1}{\sqrt{\{f(x_1)\}}} + \frac{x_2^{m-2} dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \frac{x_n^{m-3} dx_m}{\sqrt{\{f(x_m)\}}} = 0
\end{array}\right\} \ldots (2),$$

where $f(x)$ denotes a rational and integral function of x of the degree $2m-1$. If the quantities x_1, $x_2 \ldots x_m$, which satisfy this system, are regarded as functions of an independent variable t, we can put

$$\frac{dx_1}{dt} = \frac{\sqrt{\{f(x_1)\}}}{\phi'(x_1)}, \quad \frac{dx_2}{dt} = \frac{\sqrt{\{f(x_2)\}}}{\phi'(x_2)}, \text{ and so on;}$$

and if $\alpha_1, \alpha_2 \ldots \alpha_{2m-1}$ are the roots of $f(x)$ these equations can be transformed into

$$\frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{dx_1}{dt} = \frac{\sqrt{\{\psi(x_1)\}}}{\phi'(x_1)}, \quad \frac{1}{\sqrt{(\alpha_r - x_2)}} \frac{dx_2}{dt} = \frac{\sqrt{\{\psi(x_2)\}}}{\phi'(x_2)}, \ldots,$$

where $\psi(x)$ is a rational and integral function of x of the degree $2m - 2$. Whence, by differentiation with respect to t, we find

$$\frac{d}{dt}\left[\frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{dx_1}{dt}\right] = \frac{\sqrt{\{(\alpha_r - x_1)\,\psi(x_1)\}}}{\phi'(x_1)} \frac{d}{dx_1} \frac{\sqrt{\{\psi(x_1)\}}}{\phi'(x_1)}$$
$$+ \Sigma \frac{\sqrt{\{\alpha_r - x_2)\,\psi(x_1)\,\psi(x_2)\}}}{\phi'(x_2)} \frac{d}{dx_2} \frac{1}{\phi'(x_1)},$$

the latter summation extending from x_2 to x_m; but we have

$$\frac{d}{dx_2} \frac{1}{\phi'(x_1)} = \frac{1}{(x_1 - x_2)\,\phi'(x_1)},$$

whence $\dfrac{1}{\sqrt{(\alpha_r - x_1)}} \dfrac{d}{dt}\left[\dfrac{1}{\sqrt{\{(\alpha_r - x_1)\}}} \dfrac{dx_1}{dt}\right]$

$$= \frac{1}{2} \frac{d}{dx_1} \frac{\psi(x_1)}{\{\phi'(x_1)\}^2} + \Sigma \frac{\sqrt{\{\psi(x_1)\,\psi(x_2)\}}}{(x_1 - x_2)\,\phi'(x_1)\,\phi'(x_2)} \frac{\alpha_r - x_2}{\sqrt{\{(\alpha_r - x_1)(\alpha_r - x_2)\}}}.$$

Similarly $\dfrac{1}{\sqrt{(\alpha_r - x_2)}} \dfrac{d}{dt}\left[\dfrac{1}{\sqrt{(\alpha_r - x_2)}} \dfrac{dx_2}{dt}\right]$

$$= \frac{1}{2} \frac{d}{dx_2} \frac{\psi(x_2)}{\{\phi'(x_2)\}^2} + \Sigma \frac{\sqrt{\{\psi(x_1)\,\psi(x_2)\}}}{(x_2 - x_1)\,\phi'(x_1)\,\phi'(x_2)} \frac{\alpha_r - x_1}{\sqrt{\{(\alpha_r - x_1)(\alpha_r - x_2)\}}},$$

this last summation extending to $x_1, x_3 \ldots x_m$ *i.e.* to all with the exception of x_2. Hence, taking the sum of all m similar equations,

$$\Sigma \frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{d}{dt}\left[\frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{dx_1}{dt}\right]$$
$$= \frac{1}{2} \Sigma \frac{d}{dx_1} \frac{\psi(x_1)}{\{\phi'(x_1)\}^2} + \Sigma \frac{\sqrt{\{\psi(x_1)\,\psi(x_2)\}}}{\phi'(x_1)\,\phi'(x_2)} \frac{1}{\sqrt{\{(\alpha_r - x_1)(\alpha_r - x_2)\}}}.$$

But $\psi(x)$ is of the degree $2m-2$ in x, so that we have

$$\Sigma \frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{d}{dt}\left\{\frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{dx_1}{dt}\right\}$$

$$= \Sigma \frac{\sqrt{\{\psi(x_1)\,\psi(x_2)\}}}{\phi'(x_1)\,\phi'(x_2)} \frac{1}{\sqrt{(\alpha_r - x_1)(\alpha_r - x_2)}} \ldots\ldots(3).$$

Now

$$\frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{d}{dt}\left\{\frac{1}{\sqrt{(\alpha_r - x_1)}} \frac{dx_1}{dt}\right\} = \frac{1}{\alpha_r - x_1} \frac{d^2x_1}{dt^2} + \frac{1}{2} \frac{1}{(\alpha_r - x_1)^2} \frac{dx_1^2}{dt^2},$$

and $\Sigma \dfrac{\sqrt{\{\psi(x_1)\,\psi(x_2)\}}}{\phi'(x_1)\,\phi'(x_2)} \dfrac{1}{\sqrt{\{(\alpha_r - x_1)(\alpha_r - x_2)\}}}$

$$= \Sigma \frac{1}{(\alpha_r - x_1)(\alpha_r - x_2)} \frac{dx_1}{dt} \frac{dx_2}{dt}$$

$$= \frac{1}{2}\left\{\frac{\left(\frac{dx_1}{dt}\right)}{\alpha_r - x_1} + \frac{\left(\frac{dx_2}{dt}\right)}{\alpha_r - x_2} + \ldots + \frac{\left(\frac{dx_m}{dt}\right)}{\alpha_r - x_m}\right\}^2$$

$$- \frac{1}{2}\left\{\frac{\left(\frac{dx_1}{dt}\right)^2}{(\alpha_r - x_1)^2} + \frac{\left(\frac{dx_2}{dt}\right)^2}{(\alpha_r - x_2)^2} + \ldots + \frac{\left(\frac{dx_m}{dt}\right)^2}{(\alpha_r - x_m)^2}\right\},$$

so that equation (3) is transformed into the following:

$$\frac{1}{2}\left\{\frac{\left(\frac{dx_1}{dt}\right)}{\alpha_r - x_1} + \frac{\left(\frac{dx_2}{dt}\right)}{\alpha_r - x_2} + \ldots + \frac{\left(\frac{dx_m}{dt}\right)}{\alpha_r - x_m}\right\}^2$$

$$- \left\{\frac{\left(\frac{dx_1}{dt}\right)^2}{(\alpha_r - x_1)^2} + \frac{\left(\frac{dx_2}{dt}\right)^2}{(\alpha_r - x_2)^2} + \ldots + \frac{\left(\frac{dx_m}{dt}\right)^2}{(\alpha_r - x_m)^2}\right\}$$

$$- \left\{\frac{\frac{d^2x_1}{dt^2}}{\alpha_r - x_1} + \frac{\frac{d^2x_2}{dt^2}}{\alpha_r - x_2} + \ldots + \frac{\frac{d^2x_m}{dt^2}}{\alpha_r - x_m}\right\} = 0 \ldots\ldots(4).$$

Assume now

$$\sqrt{\{\phi(\alpha_r)\}} = \sqrt{\{(\alpha_r - x_1)(\alpha_r - x_2)\ldots(\alpha_r - x_m)\}} = e^u,$$

which leads to

$$\frac{d^2\sqrt{\{\phi(\alpha_r)\}}}{dt^2} = \sqrt{\{\phi(\alpha_r)\}}\left\{\frac{d^2u}{dt^2} + \frac{du^2}{dt^2}\right\},$$

also $$\frac{du}{dt} = -\frac{1}{2}\left\{\frac{\left(\frac{dx_1}{dt}\right)}{\alpha_r - x_1} + \frac{\left(\frac{dx_2}{dt}\right)}{\alpha_r - x_2} + \ldots + \frac{\left(\frac{dx_m}{dt}\right)}{\alpha_r - x_m}\right\},$$

$$\frac{d^2u}{dt^2} = -\frac{1}{2}\left\{\frac{\left(\frac{dx_1}{dt}\right)^2}{(a_r - x_1)^2} + \frac{\left(\frac{dx_2}{dt}\right)^2}{(a_r - x_2)^2} + \ldots + \frac{\left(\frac{dx_m}{dt}\right)^2}{(a_r - x_m)^2}\right\}$$
$$-\frac{1}{2}\left\{\frac{\frac{d^2x_1}{dt^2}}{a_r - x_1} + \frac{\frac{d^2x_2}{dt^2}}{a_r - x_2} + \ldots + \frac{\frac{d^2x_m}{dt^2}}{a_r - x_m}\right\},$$

whence, in consequence of equation (4),

$$\frac{d^2u}{dt^2} + \frac{du^2}{dt^2} = 0,$$

or

$$\frac{d^2\sqrt{\{\phi(a_r)\}}}{dt^2} = 0,$$

and by integration

$$\frac{d}{dt}\sqrt{\{\phi(a_r)\}} = A_r \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5),$$

where A_r is an arbitrary constant; whence, since $\phi(a_r)$ depends on t only through the quantities $x_1, x_2 \ldots x_m$, we have

$$-\frac{\sqrt{\{\phi(a_r)\}}}{2}\left[\frac{\sqrt{\{f(x_1)\}}}{(a_r - x_1)\,\phi'(x_1)} + \frac{\sqrt{\{f(x_2)\}}}{(a_r - x_2)\,\phi'(x_2)} + \ldots + \frac{\sqrt{\{f(x_m)\}}}{(a_r - x_m)\,\phi'(x_m)}\right] = A_r.$$

This last equation is an algebraic relation connecting the quantities $x_1, x_2 \ldots x_m$; and by taking for a_r any $(m-1)$ roots of $f(x)$, we have $(m-1)$ algebraic equations which are equivalent to the transcendental integrals of the system (2), and the equations obtained by taking for a_r the remaining roots of $f(x)$ are necessarily involved in the $(m-1)$ equations previously determined.

4. Since $\dfrac{x_1^{m-1}dx_1}{\sqrt{\{f(x_1)\}}} + \dfrac{x_2^{m-1}dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \dfrac{x_m^{m-1}dx_m}{\sqrt{\{f(x_m)\}}} = dt,$

if when $x_1 = a_1$ we suppose that $x_2, x_3 \ldots x_m$ take respectively the values $a_2, a_3 \ldots a_m$, we can put

$$t = \int_{a_1}^{x_1}\frac{x_1^{m-1}dx_1}{\sqrt{(fx_1)}} + \int_{a_2}^{x_2}\frac{x_2^{m-1}dx_2}{\sqrt{(fx_2)}} + \ldots + \int_{a_m}^{x_m}\frac{x_m^{m-1}dx_m}{\sqrt{(fx_m)}},$$

so that t vanishes for this system of values.

The equation (5) leads by integration to the following:

$$\sqrt{\{\phi(\alpha_r)\}} = A_r t + B_r,$$

and the quantities A_r, B_r can be expressed by means of $a_1, a_2 \ldots a_m$ as follows:

$$B_r = \sqrt{\{(\alpha_r - a_1)(\alpha_r - a_2)\ldots(\alpha_r - a_m)\}},$$

$$A_r = -\frac{B_r}{2}\left[\frac{\sqrt{\{f(a_1)\}}}{(\alpha_r - a_1)\,\phi'(a_1)} + \frac{\sqrt{\{f(a_2)}}{(\alpha_r - a_2)\,\phi'(a_2)} + \ldots + \frac{\sqrt{\{f(a_m)\}}}{(\alpha_r - a_m)\,\phi'(a_m)}\right].$$

If $\alpha_{r'}$ is another root of $f(x)$ we can put

$$\sqrt{\{\phi(\alpha_{r'})\}} = A_{r'} t + B_{r'};$$

so that from any two roots of $f(x)$ we can derive an integral of the form

$$A_{r'}\sqrt{\{\phi(\alpha_r)\}} - A_r\sqrt{\{\phi(\alpha_{r'})\}} = A_{r'}B_r - A_r B_{r'}.$$

5. Since $\phi(\alpha_r) = (A_r t + B_r)^2$, by taking any m roots of $f(x)$ we derive m equations linear in the sum, in the sum of products in pairs, and so on, of the quantities $x_1, x_2 \ldots x_m$; so that Σx_1, $\Sigma x_1 x_2$, &c. are all quadratic functions of t; and if $R(x)$ is a rational and integral function of x, $\frac{R(x_1)}{\phi'(x_1)} + \frac{R(x_2)}{\phi'(x_2)} + \ldots + \frac{R(x_m)}{\phi'(x_m)}$ is therefore a rational and integral function of t, so that

$$\int\frac{R(x_1)\,dx_1}{\sqrt{\{f(x_1)\}}} + \int\frac{R(x_2)\,dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \int\frac{R(x_m)\,dx_m}{\sqrt{\{f(x_m)\}}}$$

is a rational and integral function of t. Thus since

$$\frac{x_1^{\,m}}{\phi'(x_1)} + \frac{x_2^{\,m}}{\phi'(x_2)} + \ldots + \frac{x_m^{\,m}}{\phi'(x_m)} = x_1 + x_2 + \ldots + x_m,$$

we find

$$\int\frac{x_1^{\,m}dx_1}{\sqrt{\{f(x_1)\}}} + \int\frac{x_2^{\,m}dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \int\frac{x_m^{\,m}dx_m}{\sqrt{\{f(x_m)\}}} = \int(x_1 + x_2 + \ldots + x_m)\,dt.$$

6. For the sake of precision of our ideas we shall suppose that the roots of $f(x)$ are all real, and that the quantities $x_1, x_2 \ldots x_m$ occupy an interval terminated by two roots between which none of the remaining roots intervenes.

It is important to observe that the equation

$$\phi(a_r) = (A_r t + B_r)^2$$

gives us, according to the roots of $f(x)$ which we select, a variety of algebraic expressions for t in terms of $x_1, x_2 \ldots x_m$, $a_1, a_2 \ldots a_m$.

7. By putting

$$\frac{1}{1+nx_1} \frac{dx_1}{\sqrt{\{f(x_1)\}}} + \frac{1}{1+nx_2} \frac{dx_2}{\sqrt{\{f(x_2)\}}} + \ldots + \frac{1}{1+nx_m} \frac{dx_m}{\sqrt{\{f(x_m)\}}} = Tdt,$$

we find

$$\frac{1}{(1+nx_1)} \frac{1}{\phi'(x_1)} + \frac{1}{(1+nx_2)\,\phi'(x_2)} + \frac{1}{1+nx_m} \frac{1}{\phi'(x_m)} = T.$$

Now the left-hand side of this equation is equal to

$$\frac{(-n)^{m-1}}{(1+nx_1)(1+nx_2)\ldots(1+nx_m)},$$

but from what has been already demonstrated we infer that the denominator of this fraction is a quadratic function of t, so that

$$\frac{dx_1}{1+nx_1} \frac{1}{\sqrt{f(x_1)}} + \frac{dx_2}{(1+nx_2)} \frac{1}{\sqrt{fx_2}} + \ldots + \frac{dx_m}{1+nx_m} \frac{1}{\sqrt{f(x_m)}}$$

is the differential of a logarithmic or circular function of t.

If $-\frac{1}{n}$ is a root of $f(x)$, $(1+nx_1)(1+nx_2)\ldots(1+nx_m)$ is the square of a linear function of t, and this latter differential expression becomes the differential of an algebraic function of t.

CHAPTER II.

APPLICATION OF THE GENERAL THEOREM TO THE CASE $m=2$.

8. In this case we shall assign to $f(x)$ the form

$$x(1-x)(1-k^2x),$$

whence $\alpha_1=0$, $\alpha_2=1$, $\alpha_3=\frac{1}{k^2}$: k is supposed to be <1.

To consider the differential equation

$$\frac{dx_1}{\sqrt{\{x_1(1-x_1)(1-k^2x_1)\}}}+\frac{dx_2}{\sqrt{\{x_2(1-x_2)(1-k^2x_2)\}}}=0,$$

where we suppose that x_1 and x_2 are comprised between the roots zero and unity of $f(x)$, we shall put

$$x_1=\sin^2\theta,\ x_2=\sin^2\phi.$$

By these substitutions the above differential equation is transformed into

$$\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}+\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}=0.$$

To apply the general theorem to the equation in this form we shall put $\Delta(\theta)=\sqrt{(1-k^2\sin^2\theta)}$, whence

$$\frac{d\theta}{dt}=\frac{\Delta(\theta)}{\sin^2\theta-\sin^2\phi},\ \frac{d\phi}{dt}=\frac{\Delta(\phi)}{\sin^2\phi-\sin^2\theta},$$

and since

$$\sqrt{\{\phi(\alpha_1)\}}=\sin\theta\sin\phi,\ \sqrt{\{\phi(\alpha_2)\}}=\cos\theta\cos\phi,$$

$$\sqrt{\{\phi(\alpha_3)\}}=\frac{\Delta(\theta)\,\Delta(\phi)}{k^2},$$

we deduce from the supposition that when $\theta=0$, ϕ becomes σ, the following equations, in consequence of (5),

$$\frac{d}{dt}(\sin\theta\sin\phi) = \frac{\sin\phi\cos\theta\Delta(\theta)-\sin\theta\cos\phi\Delta(\phi)}{\sin^2\theta-\sin^2\phi} = -\frac{1}{\sin\sigma},$$

$$\frac{d}{dt}(\cos\theta\cos\phi) = \frac{\cos\theta\sin\phi\Delta(\phi)-\cos\phi\sin\theta\Delta(\theta)}{\sin^2\theta-\sin^2\phi} = -\frac{\Delta(\sigma)}{\sin\sigma},$$

$$\frac{d}{dt}\left\{\frac{\Delta(\theta)\,\Delta(\phi)}{k^2}\right\} = \frac{\sin\phi\cos\phi\Delta(\theta)-\sin\theta\cos\theta\Delta(\phi)}{\sin^2\theta-\sin^2\phi} = -\cot\sigma.$$

If we put $F(k,\theta)=\int_0^\theta \frac{d\theta}{\Delta(\theta)}$, these three equations are different forms of the same transcendental equation

$$F(k,\theta)+F(k,\phi)=F(k,\sigma).$$

9. Let $G(k,\theta)=\int_0^\theta \frac{\sin^2\theta d\theta}{\Delta(\theta)}$, then

$$G(k,\theta)+G(k,\phi)-G(k,\sigma)=t,$$

and when $\theta=0$, t vanishes also. By integration with respect to t, and remembering that when $\theta=0$, $\phi=\sigma$, we deduce from the equations given in the previous section the following:

$$\sin\theta\sin\phi\sin\sigma=-t,$$

$$\cos\theta\cos\phi=-\frac{\Delta(\sigma)}{\sin\sigma}t+\cos\sigma,$$

$$\Delta(\theta)\,\Delta(\phi)=-k^2\cot\sigma t+\Delta(\sigma).$$

By substituting in the last two of these equations the value of t derived from the first, we find

$$\cos\sigma=\cos\theta\cos\phi-\sin\theta\sin\phi\Delta(\sigma),$$

$$\Delta(\theta)\,\Delta(\phi)=\Delta(\sigma)+k^2\sin\theta\sin\phi\cos\sigma,$$

whence we deduce easily

$$\Delta(\theta)\,\Delta(\phi)\,\Delta(\sigma)=1-k^2+k^2\cos\theta\cos\phi\cos\sigma,$$

$$\sin\sigma=\frac{\sin\theta\cos\phi\Delta(\phi)+\sin\phi\cos\theta\Delta(\theta)}{1-k^2\sin^2\theta\sin^2\phi}.$$

As the transcendental equation is reproduced by interchanging θ and σ, and changing the sign of ϕ, the same processes applied to the algebraic forms will give rise to the same or equivalent results. Hence

$$\cos\theta = \cos\sigma \cos\phi + \sin\sigma \sin\phi \Delta(\theta),$$
$$\Delta(\sigma)\Delta(\phi) = \Delta(\theta) - k^2 \sin\sigma \sin\phi \cos\theta,$$

either of which is equivalent to any of the equations already given, and derivable from it by actual calculation.

10. If $$E(k, \theta) = \int_0^\theta \Delta(\theta)\, d\theta,$$

we find $E(k, \theta) + E(k, \phi) - E(k, \sigma) = k^2 \sin\theta \sin\phi \sin\sigma$; $E(k, \theta)$ is the well-known expression for the arc of an ellipse (counted from the extremity of the lesser axis) whose major semi-axis is unity, and whose eccentricity is k.

11. $$\frac{1}{(1+n\sin^2\theta)}\frac{d\theta}{\Delta(\theta)} + \frac{1}{(1+n\sin^2\phi)}\frac{d\phi}{\Delta(\phi)}$$
$$= -\frac{ndt}{1+n(\sin^2\theta+\sin^2\phi)+n^2\sin^2\theta\sin^2\phi},$$

and since $$\sin^2\theta \sin^2\phi = \frac{t^2}{\sin^2\sigma},$$

we find $\sin^2\theta + \sin^2\phi = \sin^2\sigma + 2\cot\sigma\Delta(\sigma)\, t + k^2t^2$,

so that the above differential expression becomes

$$-\frac{ndt}{1+n\sin^2\sigma + 2n\cot\sigma\Delta(\sigma)\, t + \left(nk^2 + \dfrac{n^2}{\sin^2\sigma}\right)t^2};$$

and if we put $\Pi(n, k, \theta) = \int_0^\theta \dfrac{1}{1+n\sin^2\theta}\dfrac{d\theta}{\Delta(\theta)}$,

we find, by integration,

$$\Pi(n, k, \theta) + \Pi(n, k, \phi) - \Pi(n, k, \sigma)$$
$$= \sqrt{\left\{\frac{n}{(n+1)(n+k^2)}\right\}} \tan^{-1}\frac{\sqrt{\{n(n+1)(n+k^2)\}}\sin\theta\sin\phi\sin\sigma}{1+n\sin^2\sigma - n\sin\theta\sin\phi\cos\sigma\Delta(\sigma)}$$
$$= \sqrt{\left\{\frac{n}{(n+1)(n+k^2)}\right\}} \tan^{-1}\frac{\sqrt{\{n(n+1)(n+k^2)\}}\sin\theta\sin\phi\sin\sigma}{1+n-n\cos\theta\cos\phi\cos\sigma}.$$

The circular function becomes logarithmic if the quantity under the radical is negative.

If
$$n=-k^2,$$
$$\Pi(-k^2, k, \theta)+\Pi(-k^2, k, \phi)-\Pi(-k^2, k, \sigma)$$
$$=-\frac{k^2 \sin\theta \sin\phi \sin\sigma}{\Delta(\theta)\,\Delta(\phi)\,\Delta(\sigma)}.$$

12. The functions $F(k, \theta)$, $E(k, \theta)$, $\Pi(n, k, \theta)$ are known in analysis under the names of elliptic functions of the first, second, and third kind respectively. θ is called the amplitude of the function, k its modulus, and the additional element n which enters into the third kind is called the parameter. If the amplitude is a right angle, the function is then called *complete.*

13. *Jacobi's notation.* If $u=F(k, \theta)$, then θ is denoted by the symbol $am(u)$. Thus if
$$v=F(k, \phi) \text{ and } F(k, \theta)+F(k, \phi)=F(k, \sigma),$$
then $\sigma=am(u+v)$ and the value for $\sin\sigma$ given in section (9) is expressed as follows:

$$\sin am(u+v)$$
$$=\frac{\sin am(u) \cos am(v)\, \Delta am(v)+\sin am(v) \cos am(u)\, \Delta am(u)}{1-k^2 \sin^2 am(u) \sin^2 am(v)}.$$

If $\sigma=\frac{1}{2}\pi$, $F(k, \sigma)$ is denoted by K; the functions u, v, connected by the equation $u+v=K$, are called complementary functions; the amplitude of v or of $K-u$ is called the co-amplitude of u, and denoted by the symbol $coam\,(u)$; and the following equations are easily deduced from those already given:

$$\sin coam(u)=\frac{\cos am(u)}{\Delta am(u)}, \quad \cos coam(u)=\frac{\sqrt{(1-k^2)}\sin am(u)}{\Delta am(u)},$$
$$\tan am(u) \tan coam(u)=\frac{1}{\sqrt{(1-k^2)}},$$
$$\Delta am(u)\, \Delta coam(u)=\sqrt{(1-k^2)}.$$

The comparison of elliptic functions of the second kind is written under the following form:

$$Eam(u)+Eam(v)-Eam(u+v)$$
$$=k^2 \sin am(u) \sin am(v) \sin am(u+v).$$

CHAPTER III.

APPLICATION OF THE GENERAL THEOREM TO THE CASE $m = 3$.

14. In this case we shall assume

$$f(x) = x\,(1 - x)\,(1 - \kappa^2 x)\,(1 - \kappa'^2 x)\,(1 - \kappa''^2 x),$$

and shall suppose that κ^2, κ'^2, κ''^2 are all unequal and greater than zero, and less than unity. The roots of $f(x)$ are then $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = \frac{1}{\kappa^2}$, $\alpha_4 = \frac{1}{\kappa'^2}$, $\alpha_5 = \frac{1}{\kappa''^2}$. I propose to determine the algebraic forms of the integrals of the simultaneous differential equations

$$\frac{dx_1}{\sqrt{\{f(x_1)\}}} + \frac{dx_2}{\sqrt{\{f(x_2)\}}} + \frac{dx_3}{\sqrt{\{f(x_3)\}}} = 0,$$

$$\frac{x_1 dx_1}{\sqrt{\{f(x_1)\}}} + \frac{x_2 dx_2}{\sqrt{\{f(x_2)\}}} + \frac{x_3 dx_3}{\sqrt{\{f(x_3)\}}} = 0,$$

when x_1, x_2, x_3 are situated between the roots zero and unity of $f(x)$. Let us put $x_1 = \sin^2\phi$, $x_2 = \sin^2\psi$, $x_3 = \sin^2\chi$,

$$\Delta\,(\phi) = \sqrt{\{(1 - \kappa^2 \sin^2\phi)\,(1 - \kappa'^2 \sin^2\phi)\,(1 - \kappa''^2 \sin^2\phi)\}},$$

and the above system is transformed into the following

$$\frac{d\phi}{\Delta\,(\phi)} + \frac{d\psi}{\Delta\,(\psi)} + \frac{d\chi}{\Delta\,(\chi)} = 0,$$

$$\frac{\sin^2\phi d\phi}{\Delta\,(\phi)} + \frac{\sin^2\psi d\psi}{\Delta\,(\psi)} + \frac{\sin^2\chi d\chi}{\Delta\,(\chi)} = 0.$$

15. Let us designate $\int_0^\phi \frac{d\phi}{\Delta\,(\phi)}$ by the symbol $L\,(\phi)$, or by $L\,(\kappa,\ \kappa',\ \kappa'',\ \phi)$ if we wish to put in evidence the elements

κ, κ', κ'', and let $M(\phi)$ or $M(\kappa, \kappa', \kappa'', \phi)$ denote $\int_0^\phi \frac{\sin^2\phi d\phi}{\Delta(\phi)}$. If when $\chi=0$ we suppose that ϕ and ψ become respectively σ and τ, we are led at once by integration of the above system to the transcendental equations

$$\left.\begin{aligned} L(\phi)+L(\psi)+L(\chi)-L(\sigma)-L(\tau)=0 \\ M(\phi)+M(\psi)+M(\chi)-M(\sigma)-M(\tau)=0 \end{aligned}\right\} \ldots\ldots (6).$$

16. Since $\sqrt{\{-\phi(\alpha_1)\}} = \sin\phi \sin\psi \sin\chi$,

and $\sqrt{\{\phi(\alpha_2)\}} = \cos\phi \cos\psi \cos\chi$,

if we put

$$\frac{d\phi}{dt} = \frac{\Delta(\phi)}{(\sin^2\phi - \sin^2\psi)(\sin^2\phi - \sin^2\chi)},$$

$$\frac{d\psi}{dt} = \frac{\Delta(\psi)}{(\sin^2\psi - \sin^2\chi)(\sin^2\psi - \sin^2\phi)},$$

$$\frac{d\chi}{dt} = \frac{\Delta(\chi)}{(\sin^2\chi - \sin^2\phi)(\sin^2\chi - \sin^2\psi)},$$

the general theorem leads to a pair of algebraical equations equivalent to the transcendental system (6), by equating the differential coefficients with respect to t of $\sin\phi \sin\psi \sin\chi$ and of $\cos\phi \cos\psi \cos\chi$ to the values which they assume on substituting in them σ, τ, 0 for ϕ, ψ, χ respectively. By this process we form the algebraic system

$$\left.\begin{aligned} &\frac{\sin\psi \sin\chi \cos\phi\Delta(\phi)}{(\sin^2\phi - \sin^2\psi)(\sin^2\phi - \sin^2\chi)} + \frac{\sin\chi \sin\phi \cos\psi\Delta(\psi)}{(\sin^2\psi - \sin^2\chi)(\sin^2\psi - \sin^2\phi)} \\ &+ \frac{\sin\phi \sin\psi \cos\chi\Delta(\chi)}{(\sin^2\chi - \sin^2\phi)(\sin^2\chi - \sin^2\psi)} = \frac{1}{\sin\sigma \sin\tau}, \\ &\frac{\sin\phi \cos\psi \cos\chi\Delta(\phi)}{(\sin^2\phi - \sin^2\psi)(\sin^2\phi - \sin^2\chi)} + \frac{\sin\psi \cos\chi \cos\phi\Delta(\psi)}{(\sin^2\psi - \sin^2\chi)(\sin^2\psi - \sin^2\phi)} \\ &+ \frac{\sin\chi \cos\phi \cos\psi\Delta(\chi)}{(\sin^2\chi - \sin^2\phi)(\sin^2\chi - \sin^2\psi)} = \frac{\sin\tau \cos\tau\Delta(\sigma) - \sin\sigma \cos\sigma\Delta(\tau)}{\sin\sigma \sin\tau (\sin^2\sigma - \sin^2\tau)} \end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and these equations contain the entire theory of the addition of the first class of hyperelliptic integrals.

17. Let $N(\phi)$ or $N(\kappa, \kappa', \kappa'', \phi)$ denote $\int_0^\phi \frac{\sin^4\phi\, d\phi}{\Delta(\phi)}$; if the transcendental system (6) holds, then

$$N(\phi)+N(\psi)+N(\chi)-N(\sigma)-N(\tau)=t.$$

18. Since $\frac{d}{dt}(\sin\phi \sin\psi \sin\chi)=\frac{1}{\sin\sigma \sin\tau}$,

$$\frac{d}{dt}(\cos\phi \cos\psi \cos\chi)=\frac{\sin\sigma \cos\sigma\Delta(\tau)-\sin\tau \cos\tau\Delta(\sigma)}{\sin\sigma \sin\tau(\sin^2\sigma-\sin^2\tau)},$$

by integration with respect to t and remembering that this quantity vanishes for $\chi=0$, we deduce, on putting

$$\nabla(\sigma, \tau)=\frac{\sin\sigma \cos\sigma\Delta(\tau)-\sin\tau \cos\tau\Delta(\sigma)}{\sin^2\sigma-\sin^2\tau},$$

$$t=\sin\phi \sin\psi \sin\chi \sin\sigma \sin\tau$$

$$\cos\phi \cos\psi \cos\chi=\frac{t\nabla(\sigma, \tau)}{\sin\sigma \sin\tau}+\cos\sigma \cos\tau,$$

and by substituting in the last of these equations the value of t derived from the former, we find

$$\cos\sigma \cos\tau=\cos\phi \cos\psi \cos\chi-\sin\phi \sin\psi \sin\chi\nabla(\sigma, \tau)\ldots(8).$$

The transcendental system (6) is unaltered if we interchange ϕ and σ, ψ and τ, and at the same time change the sign of χ; on treating in a similar way the equation (8), we arrive at the equation

$$\cos\phi \cos\psi=\cos\sigma \cos\tau \cos\chi+\sin\sigma \sin\tau \sin\chi\nabla(\phi, \psi)\ldots(9),$$

and the equations (8) (9) are equivalent to the transcendental system (6), or to the algebraic system (7), and are derivable from it by actual calculation.

The quantity $\nabla(\sigma, \tau)$ can be written also under the form

$$\frac{\cos^2\sigma\cos^2\tau(1-\kappa^2\kappa'^2\kappa''^2\sin^2\sigma\sin^2\tau)-(1-\kappa^2)(1-\kappa'^2)(1-\kappa''^2)\sin^2\sigma\sin^2\tau}{\sin\sigma \cos\sigma\Delta(\tau)+\sin\tau \cos\tau\Delta(\sigma)}.$$

19. If we eliminate χ between the equations (8), (9), there results

$$\sin^2\sigma \sin^2\tau[\nabla(\phi, \psi)]^2-\sin^2\phi \sin^2\psi[\nabla(\sigma, \tau)]^2$$
$$=\cos^2\phi \cos^2\psi-\cos^2\sigma \cos^2\tau\ \ldots\ldots\ldots(10).$$

If we multiply the first equation of the system (7) by $\cos\phi \cos\psi \cos\chi$, and the second equation of the same system by $\sin\phi \sin\psi \sin\chi$, and add the results, we derive, in consequence of (8), the following

$$\frac{\sin 2\psi \sin 2\chi \Delta(\phi)}{(\sin^2\phi - \sin^2\psi)(\sin^2\phi - \sin^2\chi)} + \frac{\sin 2\phi \sin 2\chi \Delta(\psi)}{(\sin^2\psi - \sin^2\chi)(\sin^2\psi - \sin^2\phi)}$$

$$+ \frac{\sin 2\phi \sin 2\psi \Delta(\chi)}{(\sin^2\chi - \sin^2\phi)(\sin^2\chi - \sin^2\psi)} = 4 \cot\sigma \cot\tau.$$

If $\chi = \frac{1}{2}\pi$, this equation gives

$$\tan\phi \tan\psi \tan\sigma \tan\tau = \frac{1}{\sqrt{\{(1-\kappa^2)(1-\kappa'^2)(1-\kappa''^2)\}}} \quad \text{...(11).}$$

By solving for $\sin\chi$, $\cos\chi$ from the equations (8), (9), we find

$$\sin\chi = \frac{\cos^2\phi \cos^2\psi - \cos^2\sigma \cos^2\tau}{\sin\phi \sin\psi \cos\sigma \cos\tau \nabla(\sigma,\tau) + \sin\sigma \sin\tau \cos\phi \cos\psi \nabla(\phi,\psi)},$$

$$\cos\chi = \frac{\sin\sigma \cos\sigma \sin\tau \cos\tau \nabla(\phi,\psi) + \sin\phi \cos\phi \sin\psi \cos\psi \nabla(\sigma,\tau)}{\sin\phi \sin\psi \cos\sigma \cos\tau \nabla(\sigma,\tau) + \sin\sigma \sin\tau \cos\phi \cos\psi \nabla(\phi,\psi)},$$

and these equations give us, by taking into account the relation (10),

$$\sin\chi = \frac{\sin\sigma \sin\tau \cos\phi \cos\psi \nabla(\phi,\psi) - \sin\phi \sin\psi \cos\sigma \cos\tau \nabla(\sigma,\tau)}{\sin^2\sigma \sin^2\tau [\nabla(\phi,\psi)]^2 + \cos^2\sigma \cos^2\tau},$$

$$\cos\chi = \frac{\cos\phi \cos\psi \cos\sigma \cos\tau + \sin\phi \sin\psi \sin\sigma \sin\tau \nabla(\phi,\psi) \nabla(\sigma,\tau)}{\sin^2\sigma \sin^2\tau [\nabla(\phi,\psi)]^2 + \cos^2\sigma \cos^2\tau}.$$

The denominator of the last expressions can be written under the equivalent form $\sin^2\phi \sin^2\psi [\nabla(\sigma,\tau)]^2 + \cos^2\phi \cos^2\psi$, and, on dividing one by the other, we find

$$\tan\chi = \frac{\tan\sigma \tan\tau \nabla(\phi,\psi) - \tan\phi \tan\psi \nabla(\sigma,\tau)}{1 + \tan\phi \tan\psi \tan\sigma \tan\tau \nabla(\phi,\psi) \nabla(\sigma,\tau)},$$

and from this equation the following are easily seen to be true:

$$\phi = \tan^{-1}\{\tan\sigma \tan\tau \nabla(\psi,\chi)\} - \tan^{-1}\{\tan\psi \tan\chi \nabla(\sigma,\tau)\},$$

$$\psi = \tan^{-1}\{\tan\sigma \tan\tau \nabla(\chi,\phi)\} - \tan^{-1}\{\tan\chi \tan\phi \nabla(\sigma,\tau)\},$$

$$\chi = \tan^{-1}\{\tan\sigma \tan\tau \nabla(\phi,\psi)\} - \tan^{-1}\{\tan\phi \tan\psi \nabla(\sigma,\tau)\},$$

and any one of these is necessarily involved in the other two.

20. If $\chi=\frac{1}{2}\pi$, the transcendental system

$$L(\phi)+L(\psi)+L(\tfrac{1}{2}\pi)-L(\sigma)-L(\tau)=0,$$
$$M(\phi)+M(\psi)+M(\tfrac{1}{2}\pi)-M(\sigma)-M(\tau)=0,$$

can be replaced by the algebraic system

$$\surd\{(1-\kappa^2)(1-\kappa'^2)(1-\kappa''^2)\}\tan\phi\tan\psi\tan\sigma\tan\tau=1,$$
$$\nabla(\phi,\psi)\,\nabla(\sigma,\tau)+\surd\{(1-\kappa^2)(1-\kappa'^2)(1-\kappa''^2)\}=0.$$

21. The following equations connecting ϕ, ψ, χ, σ, τ are easily deduced:

$$\cos\phi\cos\tau=\cos\sigma\cos\psi\cos\chi-\sin\sigma\sin\psi\sin\chi\nabla(\phi,-\tau),$$
$$\cos\psi\cos\sigma=\cos\tau\cos\phi\cos\chi-\sin\tau\sin\phi\sin\chi\nabla(\psi,-\sigma),$$

$$\frac{\sin\sigma\sin\psi\cos\chi\Delta(\chi)}{(\sin^2\chi-\sin^2\sigma)(\sin^2\chi-\sin^2\psi)}+\frac{\sin\sigma\sin\chi\cos\psi\Delta(\psi)}{(\sin^2\psi-\sin^2\sigma)(\sin^2\psi-\sin^2\chi)}$$
$$-\frac{\sin\psi\sin\chi\cos\sigma\Delta(\sigma)}{(\sin^2\sigma-\sin^2\psi)(\sin^2\sigma-\sin^2\chi)}=\frac{1}{\sin\phi\sin\tau},$$

$$\frac{\sin\sigma\sin\tau\cos\chi\Delta(\chi)}{(\sin^2\chi-\sin^2\sigma)(\sin^2\chi-\sin^2\tau)}-\frac{\sin\tau\sin\chi\cos\sigma\Delta(\sigma)}{(\sin^2\sigma-\sin^2\chi)(\sin^2\sigma-\sin^2\tau)}$$
$$-\frac{\sin\sigma\sin\chi\cos\tau\Delta(\tau)}{(\sin^2\tau-\sin^2\sigma)(\sin^2\tau-\sin^2\chi)}=\frac{1}{\sin\phi\sin\psi},$$

$$\frac{\sin2\sigma\sin2\tau\Delta(\chi)}{(\sin^2\chi-\sin^2\sigma)(\sin^2\chi-\sin^2\tau)}-\frac{\sin2\sigma\sin2\chi\Delta(\tau)}{(\sin^2\tau-\sin^2\sigma)(\sin^2\tau-\sin^2\chi)}$$
$$-\frac{\sin2\tau\sin2\chi\Delta(\sigma)}{(\sin^2\sigma-\sin^2\tau)(\sin^2\sigma-\sin^2\chi)}=4\cot\phi\cot\psi,$$

and many similar equations will readily suggest themselves.

22. The algebraic forms of the transcendental system (6) derived from the roots $\frac{1}{\kappa^2}$, $\frac{1}{\kappa'^2}$, $\frac{1}{\kappa''^2}$ of $f(x)$ are really involved in the equations already given: but their deduction by this method would require troublesome and difficult calculations; and the readiness with which they are derived from the general method constitutes one of the greatest advantages attending the form under which Jacobi has pre-

sented Abel's theorem. Since $\sqrt{\{\phi(\alpha_3)\}} = \dfrac{\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}}}{\kappa^3}$, if we equate $\dfrac{d}{dt}\left[\dfrac{\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}}}{\kappa^3}\right]$ to the value which it assumes when ϕ, ψ, χ are replaced respectively by $\sigma, \tau, 0$, we obtain an equation which, combined with any of the equations already given, forms a system equivalent to the transcendental system (6). In this manner we arrive at the following equation:

$$\frac{\sin\phi\cos\phi\sqrt{\{(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)\}}}{(\sin^2\phi-\sin^2\psi)(\sin^2\phi-\sin^2\chi)}$$

$$+\frac{\sin\psi\cos\psi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\}}}{(\sin^2\psi-\sin^2\phi)(\sin^2\psi-\sin^2\chi)}$$

$$+\frac{\sin\chi\cos\chi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa'^2\sin^2\chi)(1-\kappa''^2\sin^2\chi)\}}}{(\sin^2\chi-\sin^2\phi)(\sin^2\chi-\sin^2\psi)}$$

$$=\frac{\sin\tau\cos\sigma\sqrt{\{(1-\kappa^2\sin^2\tau)(1-\kappa'^2\sin^2\sigma)(1-\kappa''^2\sin^2\sigma)\}}-\sin\sigma\cos\tau\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)(1-\kappa''^2\sin^2\tau)\}}}{\sin\sigma\sin\tau(\sin^2\sigma-\sin^2\tau)}$$

$$\dots\dots\dots\dots\dots\dots(12).$$

Since then $\dfrac{d}{dt}\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\tau)\}}$

$$=\kappa^2\left[\frac{\sin\sigma\cos\tau\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)(1-\kappa''^2\sin^2\tau)\}}-\sin\tau\cos\sigma\sqrt{\{(1-\kappa^2\sin^2\tau)(1-\kappa'^2\sin^2\sigma)(1-\kappa''^2\sin^2\sigma)\}}}{\sin\sigma\sin\tau(\sin^2\sigma-\sin^2\tau)}\right],$$

if we integrate this equation, determining the constant from the condition that σ, τ, 0 are simultaneous values of ϕ, ψ, χ, and introducing the value $\sin\phi \sin\psi \sin\chi \sin\sigma \sin\tau$ already found for t, there results

$$\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}}-\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}$$

$$=\kappa^2\sin\phi\sin\psi\sin\chi$$

$$\times\left[\frac{\sin\sigma\cos\tau\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)(1-\kappa''^2\sin^2\tau)\}}-\sin\tau\cos\sigma\sqrt{\{(1-\kappa^2\sin^2\tau)(1-\kappa'^2\sin^2\sigma)(1-\kappa''^2\sin^2\sigma)\}}}{\sin^2\sigma-\sin^2\tau}\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

By interchanging κ, κ' and κ, κ'' successively in this equation, we arrive at the integrals derived from the remaining roots of $f(x)$.

23. If we multiply both sides of (12) by $\kappa^2\sin\phi\sin\psi\sin\chi$, we find, in consequence of (13),

$$\frac{\kappa^2\sin^2\phi\sin\psi\sin\chi\cos\phi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)\}}}{(\sin^2\phi-\sin^2\psi)(\sin^2\phi-\sin^2\chi)}$$

$$+\frac{\kappa^2\sin^2\psi\sin\phi\sin\chi\cos\psi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\}}}{(\sin^2\psi-\sin^2\phi)(\sin^2\psi-\sin^2\chi)}$$

$$+\frac{\kappa^2\sin^2\chi\sin\phi\sin\psi\cos\chi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa'^2\sin^2\chi)(1-\kappa''^2\sin^2\chi)\}}}{(\sin^2\chi-\sin^2\phi)(\sin^2\chi-\sin^2\psi)}$$

$$=\frac{\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}-\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}}}{\sin\sigma\sin\tau}.$$

Now let us multiply both sides of the first equation of the system (7) by

$$\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}},$$

and we find

$$\frac{\sin\psi\sin\chi\cos\phi\,(1-\kappa^2\sin^2\phi)\sqrt{\{(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)\}}}{(\sin^2\phi-\sin^2\psi)(\sin^2\phi-\sin^2\chi)}$$
$$+\frac{\sin\phi\sin\chi\cos\psi\,(1-\kappa^2\sin^2\psi)\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\}}}{(\sin^2\psi-\sin^2\phi)(\sin^2\psi-\sin^2\chi)}$$
$$+\frac{\sin\phi\sin\psi\cos\chi\,(1-\kappa^2\sin^2\chi)\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa'^2\sin^2\chi)(1-\kappa''^2\sin^2\chi)\}}}{(\sin^2\chi-\sin^2\phi)(\sin^2\chi-\sin^2\psi)}$$
$$=\frac{\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}}}{\sin\sigma\sin\tau},$$

so that

$$\frac{\sin\psi\sin\chi\cos\phi\sqrt{\{(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)\}}}{(\sin^2\phi-\sin^2\psi)(\sin^2\phi-\sin^2\chi)}$$
$$+\frac{\sin\phi\sin\chi\cos\psi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\chi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\}}}{(\sin^2\psi-\sin^2\phi)(\sin^2\psi-\sin^2\chi)}$$
$$+\frac{\sin\phi\sin\psi\cos\chi\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa'^2\sin^2\chi)(1-\kappa''^2\sin^2\chi)\}}}{(\sin^2\chi-\sin^2\phi)(\sin^2\chi-\sin^2\psi)}$$
$$=\frac{\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}}{\sin\sigma\sin\tau}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

24. It follows, from what has been remarked in section 5, that any rational and integral symmetric function of $\sin^2\phi$, $\sin^2\psi$, $\sin^2\chi$ is a rational and integral function of t, and I now proceed to determine the actual values of $\sin^2\phi+\sin^2\psi+\sin^2\chi$ and of $\sin^2\phi\sin^2\psi+\sin^2\phi\sin^2\chi+\sin^2\psi\sin^2\chi$ when expressed in terms of t.

Let us put $\sin^2\phi+\sin^2\psi+\sin^2\chi=g$, $\sin^2\phi\sin^2\psi+\sin^2\phi\sin^2\chi+\sin^2\psi\sin^2\chi=h$,

$$\frac{\sin\sigma\cos\tau\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)(1-\kappa''^2\sin^2\tau)\}}-\sin\tau\cos\sigma\sqrt{\{(1-\kappa^2\sin^2\tau)(1-\kappa'^2\sin^2\sigma)(1-\kappa''^2\sin^2\sigma)\}}}{\sin^2\sigma-\sin^2\tau}=A,$$

$$\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}=B.$$

Equation (13) gives

$$\kappa^2g-\kappa^4h=1-B^2-\frac{2\kappa^2ABt}{\sin\sigma\sin\tau}-\frac{\kappa^4(A^2+\kappa^2)t^2}{\sin^2\sigma\sin^2\tau},$$

and equation (8) leads to

$$g-h=1-\cos^2\sigma\cos^2\tau-\frac{2\cos\sigma\cos\tau}{\sin\sigma\sin\tau}\nabla(\sigma,\tau)t-\left[\frac{1+\{\nabla(\sigma,\tau)\}^2}{\sin^2\sigma\sin^2\tau}\right]t^2.$$

Solving for g and h from these equations, we find

$$g=\sin^2\sigma+\sin^2\tau-2\left\{\frac{\sin\sigma\cos\tau\Delta(\tau)-\sin\tau\cos\sigma\Delta(\sigma)}{\sin\sigma\sin\tau(\sin^2\sigma-\sin^2\tau)}\right\}t+\kappa^2\kappa'^2\kappa''^2t^2,$$

$$h=\sin^2\sigma\sin^2\tau-2\left\{\frac{\sin^3\sigma\cos\tau\Delta(\tau)-\sin^3\tau\cos\sigma\Delta(\sigma)}{\sin\sigma\sin\tau(\sin^2\sigma-\sin^2\tau)}\right\}t+\left[\frac{1+\kappa^2\kappa'^2\kappa''^2\sin^2\sigma\sin^2\tau+\{\nabla(\sigma,\tau)\}^2}{\sin^2\sigma\sin^2\tau}\right]t^2,$$

and if we put $\nabla'(\sigma, \tau) = \dfrac{\sin\sigma \cos\tau \Delta(\tau) - \sin\tau \cos\sigma \Delta(\sigma)}{\sin^2\sigma - \sin^2\tau}$, these values become

$$g = \sin^2\sigma + \sin^2\tau - \frac{2\nabla'(\sigma, \tau)}{\sin\sigma \sin\tau} t + \kappa^2\kappa'^2\kappa''^2 t^2,$$

$$h = \sin^2\sigma \sin^2\tau - 2\left\{\frac{\nabla'(\sigma, \tau) - \cos\sigma \cos\tau \nabla(\sigma, \tau)}{\sin\sigma \sin\tau}\right\} t + \left[\frac{1 + \kappa^2\kappa'^2\kappa''^2 \sin^2\sigma \sin^2\tau + \{\nabla(\sigma, \tau)\}^2}{\sin^2\sigma \sin^2\tau}\right] t^2.$$

25. If $\quad \dfrac{d\phi}{\Delta(\phi)} + \dfrac{d\psi}{\Delta(\psi)} + \dfrac{d\chi}{\Delta(\chi)} = 0, \quad \dfrac{\sin^2\phi d\phi}{\Delta(\phi)} + \dfrac{\sin^2\psi d\psi}{\Delta(\psi)} + \dfrac{\sin^2\chi d\chi}{\Delta(\chi)} = 0,$

we have

$$\frac{1}{(1+n\sin^2\phi)} \cdot \frac{d\phi}{\Delta(\phi)} + \frac{1}{(1+n\sin^2\psi)} \cdot \frac{d\psi}{\Delta(\psi)} + \frac{1}{(1+n\sin^2\chi)} \cdot \frac{d\chi}{\Delta(\chi)} = \frac{n^2 dt}{(1+n\sin^2\phi)(1+n\sin^2\psi)(1+n\sin^2\chi)},$$

and if we put the denominator of the right-hand side of this equation under the form $a + 2bt + ct^2$, we find, in consequence of the values deduced for g and h,

$$a = (1 + n\sin^2\sigma)(1 + n\sin^2\tau),$$

$$b = -n\left\{\frac{(1+n)\nabla'(\sigma, \tau) - n\cos\sigma \cos\tau \nabla(\sigma, \tau)}{\sin\sigma \sin\tau}\right\}, \quad c = n\left[(1+n)\left(\kappa^2\kappa'^2\kappa''^2 + \frac{n}{\sin^2\sigma \sin^2\tau}\right) + \frac{n\{\nabla(\sigma, \tau)\}^2}{\sin^2\sigma \sin^2\tau}\right],$$

so that if we designate $\int_0^\phi \frac{1}{1+n\sin^2\phi}\frac{d\phi}{\Delta(\phi)}$ by $P(n, \phi)$ or by $P(n, \kappa, \kappa', \kappa'', \phi)$, we have, putting $ac-b^2=\alpha$,

$$P(n, \phi)+P(n, \psi)+P(n, \chi)-P(n, \sigma)-P(n, \tau)=\frac{n^2}{\sqrt{\alpha}}\tan^{-1}\frac{t\sqrt{\alpha}}{a+bt} \text{ ....} (15).$$

Now $\alpha=n(n+1)(1+n\sin^2\sigma)(1+n\sin^2\tau)\left(\kappa^2\kappa'^2\kappa''^2+\frac{n}{\sin^2\sigma\sin^2\tau}\right)$

$$+n^2(n+1)\left\{\frac{(n+1)[\sin^2\tau\{\Delta(\sigma)\}^2-\sin^2\sigma\{\Delta(\tau)\}^2]+n[\sin^2\sigma\cos^2\sigma\{\Delta(\tau)\}^2-\sin^2\tau\cos^2\tau\{\Delta(\sigma)\}^2]}{\sin^2\sigma\sin^2\tau(\sin^2\sigma-\sin^2\tau)}\right\},$$

but $\sin^2\sigma\cos^2\sigma\{\Delta(\tau)\}^2-\sin^2\tau\cos^2\tau\{\Delta(\sigma)\}^2$

$$=(\sin^2\sigma-\sin^2\tau)\{\cos^2\sigma\cos^2\tau(1-\kappa^2\kappa'^2\kappa''^2\sin^2\sigma\sin^2\tau)-(1-\kappa^2)(1-\kappa'^2)(1-\kappa''^2)\sin^2\sigma\sin^2\tau\},$$

and $\frac{\sin^2\tau\{\Delta(\sigma)\}^2-\sin^2\sigma\{\Delta(\tau)\}^2}{\sin^2\sigma-\sin^2\tau}=-1+(\kappa^2\kappa'^2+\kappa^2\kappa''^2+\kappa'^2\kappa''^2)\sin^2\sigma\sin^2\tau-\kappa^2\kappa'^2\kappa''^2\sin^2\sigma\sin^2\tau(\sin^2\sigma+\sin^2\tau)$,

and these relations cause σ and τ to disappear from the value of α, as they obviously ought to do, and there results

$$\alpha=n(n+1)(n+\kappa^2)(n+\kappa'^2)(n+\kappa''^2),$$

as might be readily foreseen.

26. If in equation (15) we interchange ϕ and σ, ψ and τ, and at the same time change the sign of χ, the quantity $\frac{t}{a+bt}$ changes its sign; and as these processes change the sign of t, they have no effect

on the value of $a+bt$; hence by reference to the values of a and b we deduce the following equation:

$$(1+n\sin^2\sigma)(1+n\sin^2\tau)-n\sin\phi\sin\psi\sin\chi\{(1+n)\nabla'(\sigma,\tau)-n\cos\sigma\cos\tau\nabla(\sigma,\tau)\}$$

$$=(1+n\sin^2\phi)(1+n\sin^2\psi)+n\sin\sigma\sin\tau\sin\chi\{(1+n)\nabla'(\phi,\psi)-n\cos\phi\cos\psi\nabla(\phi,\psi)\}\text{............}(16),$$

and this gives, in consequence of (8), (9),

$$\sin^2\phi+\sin^2\psi+\sin\sigma\sin\tau\sin\chi\nabla'(\phi,\psi)=\sin^2\sigma+\sin^2\tau-\sin\phi\sin\psi\sin\chi\nabla'(\sigma,\tau),$$

whence
$$\sin\chi=\frac{\sin^2\sigma+\sin^2\tau-\sin^2\phi-\sin^2\psi}{\sin\sigma\sin\tau\nabla'(\phi,\psi)+\sin\phi\sin\psi\nabla'(\sigma,\tau)}.$$

Now $\{\nabla'(\sigma,\tau)\}^2-\{\nabla(\sigma,\tau)\}^2=1-(\kappa^2\kappa'^2+\kappa^2\kappa''^2+\kappa'^2\kappa''^2)\sin^2\sigma\sin^2\tau+\kappa^2\kappa'^2\kappa''^2\sin^2\sigma\sin^2\tau(\sin^2\sigma+\sin^2\tau)$,

hence, in consequence of (10),

$$\sin^2\sigma\sin^2\tau\{\nabla'(\phi,\psi)\}^2-\sin^2\phi\sin^2\psi\{\nabla'(\sigma,\tau)\}^2=(\sin^2\sigma+\sin^2\tau-\sin^2\phi-\sin^2\psi)(1-\kappa^2\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi\sin^2\sigma\sin^2\tau),$$

so that we find
$$\sin\chi=\frac{\sin\sigma\sin\tau\nabla'(\phi,\psi)-\sin\phi\sin\psi\nabla'(\sigma,\tau)}{1-\kappa^2\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi\sin^2\sigma\sin^2\tau}.$$

27. We find easily $(1-\kappa^2)\nabla'(\sigma,\tau)+\kappa^2\cos\sigma\cos\tau\nabla(\sigma,\tau)=\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}$

$$\times\left[\frac{\sin\sigma\cos\tau\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)(1-\kappa''^2\sin^2\tau)\}}-\sin\tau\cos\sigma\sqrt{\{(1-\kappa^2\sin^2\tau)(1-\kappa'^2\sin^2\sigma)(1-\kappa''^2\sin^2\sigma)\}}}{\sin^2\sigma-\sin^2\tau}\right]$$

= (in consequence of (13))

$$\frac{\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}}{\kappa^2\sin\phi\sin\psi\sin\chi}\left[\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)\}}-\sqrt{\{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}\right],$$

thus we get the following:

$$\left.\begin{aligned}&\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}\\ &=(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)+\kappa^2\sin\phi\sin\psi\sin\chi\{(1-\kappa^2)\nabla'(\sigma,\tau)+\kappa^2\cos\sigma\cos\tau\nabla(\sigma,\tau)\}\\ &=(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)-\kappa^2\sin\sigma\sin\tau\sin\chi\{(1-\kappa^2)\nabla'(\phi,\psi)+\kappa^2\cos\phi\cos\psi\nabla(\phi,\psi)\}\end{aligned}\right\}\text{.........}(17),$$

the latter expression being got by writing $-\kappa^2$ for n in (16).

These equations necessarily involve the equations derived from them by interchanging κ, κ' and κ, κ'', and are equivalent to the transcendental system (6).

If we put $n=-\kappa^2$ in (15), we find, by applying (17),

$$\left.\begin{aligned}&P(-\kappa^2,\phi)+P(-\kappa^2,\psi)+P(-\kappa^2,\chi)-P(-\kappa^2,\sigma)-P(-\kappa^2,\tau)\\ &=\frac{\kappa^4\sin\phi\sin\psi\sin\chi\sin\sigma\sin\tau}{(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)+\kappa^2\sin\phi\sin\psi\sin\chi\{(1-\kappa^2)\nabla'(\sigma,\tau)+\kappa^2\cos\sigma\cos\tau\nabla(\sigma,\tau)\}}\\ &=\frac{\kappa^4\sin\phi\sin\psi\sin\chi\sin\sigma\sin\tau}{\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}}\end{aligned}\right\}\text{.........}(18).$$

If we put $n=-1$ in (15), we have, by similarly applying (8),

$$\begin{aligned}&P(-1,\phi)+P(-1,\psi)+P(-1,\chi)-P(-1,\sigma)-P(-1,\tau)\\ &=\frac{\sin\phi\sin\psi\sin\chi\sin\sigma\sin\tau}{\cos\sigma\cos\tau\{\cos\sigma\cos\tau+\sin\phi\sin\psi\sin\chi\nabla(\sigma,\tau)\}}\\ &=\tan\phi\tan\psi\tan\chi\tan\sigma\tan\tau.\end{aligned}$$

As in the values of

$\sin^2\phi+\sin^2\psi+\sin^2\chi$ and of $\sin^2\phi\sin^2\psi+\sin^2\phi\sin^2\chi+\sin^2\psi\sin^2\chi$ the quantities κ, κ', κ'' enter symmetrically, it appears that we might have employed for their determination equations (8), (9) along with either of the equations derived from (13), by interchanging κ and κ', or κ and κ''; which equations are thus shewn by actual calculation to be involved in equations (8), (9), (13).

28. Let us put

$$F=\kappa^2+\kappa'^2+\kappa''^2,\quad G=\kappa^2\kappa'^2+\kappa^2\kappa''^2+\kappa'^2\kappa''^2,\quad H=\kappa^2\kappa'^2\kappa''^2,$$

and we find

$$\frac{d}{d\theta}.\sin^{2m-1}\theta\cos\theta\Delta(\theta)$$

$$=\frac{\sin^{2m-2}\theta}{\Delta(\theta)}\{(2m-1)-2m(1+F)\sin^2\theta+(2m+1)(F+G)\sin^4\theta$$
$$-(2m+2)(G+H)\sin^6\theta+(2m+3)H\sin^8\theta\},$$

so that the integration of $\dfrac{\sin^{2m}\theta d\theta}{\Delta(\theta)}$ can be reduced to depend on the integration of

$$\frac{\sin^{2m-2}\theta d\theta}{\Delta(\theta)},\ \frac{\sin^{2m-4}\theta d\theta}{\Delta(\theta)},\ \frac{\sin^{2m-6}\theta d\theta}{\Delta(\theta)},\ \frac{\sin^{2m-8}\theta d\theta}{\Delta(\theta)},$$

and if we designate by $Q(\theta)$ $\displaystyle\int_0^\theta\frac{\sin^6\theta d\theta}{\Delta(\theta)}$, $\displaystyle\int_0^\theta\frac{\sin^{2m}\theta d\theta}{\Delta(\theta)}$ depends on the functions $L(\theta)$, $M(\theta)$, $N(\theta)$, $Q(\theta)$.

If we suppose that the system (6) holds, we have

$$Q(\phi)+Q(\psi)+Q(\chi)-Q(\sigma)-Q(\tau)=\int(\sin^2\phi+\sin^2\psi+\sin^2\chi)\,dt,$$

and by substituting for $\sin^2\phi+\sin^2\psi+\sin^2\chi$ its value already found in terms of t, the right-hand side of this equation becomes

$$(\sin^2\sigma+\sin^2\tau)\,t-\frac{\nabla'(\sigma,\tau)}{\sin\sigma\sin\tau}\,t^2+\frac{\kappa^2\kappa'^2\kappa''^2}{3}\,t^3.$$

29. If we put $N'(\theta)=\displaystyle\int_0^\theta\frac{\tan^2\theta\sin^2\theta d\theta}{\Delta(\theta)}$,

we have $\qquad P(-1,\theta)=L(\theta)+M(\theta)+N'(\theta)$ .... (19), but by differentiating $\tan\theta\Delta(\theta)$, we deduce

$$\frac{d}{d\theta}\tan\theta\Delta(\theta)=\frac{1-F\sin^2\theta+G\sin^4\theta-H\sin^6\theta}{\cos^2\theta\Delta(\theta)}$$
$$-\frac{\sin^2\theta}{\Delta(\theta)}\{F-2G\sin^2\theta+3H\sin^4\theta\},$$

whence, by integration,

$$P(-1,\theta)=\tan\theta\Delta(\theta)+2FM(\theta)-(2G+H)N(\theta)$$
$$+(F-G+H)N'(\theta)+3HQ(\theta),$$

whence, eliminating $N'(\theta)$ between this last equation and (19), we find

$$(1-\kappa^2)(1-\kappa'^2)(1-\kappa''^2)P(-1,\theta)$$
$$=(F+G-H)M(\theta)-(F-G+H)L(\theta)-(2G+H)N(\theta)$$
$$+3HQ(\theta)+\tan\theta\Delta(\theta).$$

30. We find

$$\kappa^6\frac{d}{d\theta}\cdot\left[\frac{\sin\theta\cos\theta\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)\}}}{\sqrt{(1-\kappa^2\sin^2\theta)}}\right]$$
$$=\quad\{\kappa^4(1+\kappa'^2+\kappa''^2)-\kappa^2(\kappa'^2+\kappa''^2+\kappa'^2\kappa''^2)+\kappa'^2\kappa''^2\}\frac{1}{\Delta(\theta)}$$
$$-\kappa^2\{\kappa^4(1+\kappa'^2+\kappa''^2)+\kappa^2(\kappa'^2+\kappa''^2+\kappa'^2\kappa''^2)-\kappa'^2\kappa''^2\}\frac{\sin^2\theta}{\Delta(\theta)}$$
$$+\kappa^4\{2\kappa^2(\kappa'^2+\kappa''^2+\kappa'^2\kappa''^2)+\kappa'^2\kappa''^2\}\frac{\sin^4\theta}{\Delta(\theta)}$$
$$-3\kappa^6\kappa'^2\kappa''^2\frac{\sin^6\theta}{\Delta(\theta)}-\frac{(1-\kappa^2)(\kappa^2-\kappa'^2)(\kappa^2-\kappa''^2)}{1-\kappa^2\sin^2\theta}\frac{1}{\Delta(\theta)},$$

whence $P(-\kappa^2,\theta)$ is made to depend on $L(\theta)$, $M(\theta)$, $N(\theta)$, and $Q(\theta)$ by the formula

$$(1-\kappa^2)(\kappa^2-\kappa'^2)(\kappa^2-\kappa''^2)P(-\kappa^2,\theta)$$
$$=\quad\{\kappa^4(1+\kappa'^2+\kappa''^2)-\kappa^2(\kappa'^2+\kappa''^2+\kappa'^2\kappa''^2)+\kappa'^2\kappa''^2\}L(\theta)$$
$$-\kappa^2\{\kappa^4(1+\kappa'^2+\kappa''^2)+\kappa^2(\kappa'^2+\kappa''^2+\kappa'^2\kappa''^2)-\kappa'^2\kappa''^2\}M(\theta)$$
$$+\kappa^4\{2\kappa^2(\kappa'^2+\kappa''^2+\kappa'^2\kappa''^2)+\kappa'^2\kappa''^2\}N(\theta)$$
$$-3\kappa^6\kappa'^2\kappa''^2Q(\theta)-\kappa^6\frac{\sin\theta\cos\theta\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)\}}}{\sqrt{(1-\kappa^2\sin^2\theta)}}$$
$$\dots\dots\dots\dots(20).$$

Similar formulæ for $P(-\kappa'^2, \theta)$ $P(-\kappa''^2, \theta)$ are deduced from this equation by interchanging in it successively κ and κ', κ and κ''.

31. We find also

$$\frac{1-\kappa^2}{\kappa^2} P(-\kappa^2, \theta) + \frac{1-\kappa'^2}{\kappa'^2} P(-\kappa'^2, \theta) + \frac{1-\kappa''^2}{\kappa''^2} P(-\kappa''^2, \theta) = \left(\frac{1}{\kappa^2} + \frac{1}{\kappa'^2} + \frac{1}{\kappa''^2} - 2\right) L(\theta) + M(\theta) - \frac{\sin\theta\cos\theta}{\Delta(\theta)},$$

so that if ϕ, ψ, χ, σ, τ satisfy the system (6) we derive from (18)

$$\frac{\sin\sigma\cos\sigma}{\Delta(\sigma)} + \frac{\sin\tau\cos\tau}{\Delta(\tau)} - \frac{\sin\phi\cos\phi}{\Delta(\phi)} - \frac{\sin\psi\cos\psi}{\Delta(\psi)} - \frac{\sin\chi\cos\chi}{\Delta(\chi)}$$

$$= \sin\phi\sin\psi\sin\chi\sin\sigma\sin\tau \left\{ \begin{aligned} &\frac{\kappa^2(1-\kappa^2)}{\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}} \\ &+ \frac{\kappa'^2(1-\kappa'^2)}{\sqrt{\{(1-\kappa'^2\sin^2\phi)(1-\kappa'^2\sin^2\psi)(1-\kappa'^2\sin^2\chi)(1-\kappa'^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)\}}} \\ &+ \frac{\kappa''^2(1-\kappa''^2)}{\sqrt{\{(1-\kappa''^2\sin^2\phi)(1-\kappa''^2\sin^2\psi)(1-\kappa''^2\sin^2\chi)(1-\kappa''^2\sin^2\sigma)(1-\kappa''^2\sin^2\tau)\}}} \end{aligned} \right\}.$$

We find $\quad \dfrac{d}{d\theta}\cdot\dfrac{\tan\theta}{\Delta(\theta)} = \dfrac{1}{\cos^2\theta\Delta(\theta)} + \left\{\dfrac{1}{1-\kappa^2\sin^2\theta} + \dfrac{1}{1-\kappa'^2\sin^2\theta} + \dfrac{1}{1-\kappa''^2\sin^2\theta} - 3\right\}\dfrac{1}{\Delta(\theta)},$

so that, by integration, we deduce

$$P(-1, \theta) + P(-\kappa^2, \theta) + P(-\kappa'^2, \theta) + P(-\kappa''^2, \theta) - 3L(\theta) = \frac{\tan\theta}{\Delta(\theta)},$$

and from this relation we infer, if ϕ, ψ, χ, σ, τ satisfy the system (6),

$$\frac{\tan\phi}{\Delta(\phi)} + \frac{\tan\psi}{\Delta(\psi)} + \frac{\tan\chi}{\Delta(\chi)} - \frac{\tan\sigma}{\Delta(\sigma)} - \frac{\tan\tau}{\Delta(\tau)} - \tan\phi\tan\psi\tan\chi\tan\sigma\tan\tau$$

$$= \sin\phi\sin\psi\sin\chi\sin\sigma\sin\tau\left\{\begin{aligned}&\frac{\kappa^4}{\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)(1-\kappa^2\sin^2\chi)(1-\kappa^2\sin^2\sigma)(1-\kappa^2\sin^2\tau)\}}}\\ &+\frac{\kappa'^4}{\sqrt{\{(1-\kappa'^2\sin^2\phi)(1-\kappa'^2\sin^2\psi)(1-\kappa'^2\sin^2\chi)(1-\kappa'^2\sin^2\sigma)(1-\kappa'^2\sin^2\tau)\}}}\\ &+\frac{\kappa''^4}{\sqrt{\{(1-\kappa''^2\sin^2\phi)(1-\kappa''^2\sin^2\psi)(1-\kappa''^2\sin^2\chi)(1-\kappa''^2\sin^2\sigma)(1-\kappa''^2\sin^2\tau)\}}}\end{aligned}\right\}.$$

32. In what follows it will be necessary to take into account the relative magnitudes of the quantities κ, κ', κ'', and we shall suppose that $\kappa'' > \kappa' > \kappa$; if we put $k^2 = 1-\kappa^2$, $k'^2 = 1-\kappa'^2$, $k''^2 = 1-\kappa''^2$, then $k'' < k' < k$. If we transform the differential $\frac{(\alpha+\beta\sin^2\theta)\,d\theta}{\Delta(\theta)}$, where α, β are arbitrary quantities, by the substitution $k''\tan\theta\tan\phi = 1$, we have, since

$$\frac{d\theta}{\sqrt{(1-\kappa''^2\sin^2\theta)}} = -\frac{d\phi}{\sqrt{(1-\kappa''^2\sin^2\phi)}},\quad \sin^2\theta = \frac{\cos^2\phi}{1-\kappa''^2\sin^2\phi},$$

$$\frac{(\alpha+\beta\sin^2\theta)\,d\theta}{\Delta(\theta)} = -\frac{(\alpha'+\beta'\sin^2\phi)\,d\phi}{\sqrt{\left\{\left(1-\frac{\kappa''^2-\kappa^2}{k^2}\sin^2\phi\right)\left(1-\frac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\phi\right)(1-\kappa''^2\sin^2\phi)\right\}}},$$

where $\alpha' = \frac{\alpha+\beta}{kk'}$, $\beta' = -\frac{\kappa''^2\alpha+\beta}{kk'}$; $\frac{\kappa''^2-\kappa^2}{k^2}$, $\frac{\kappa''^2-\kappa'^2}{k'^2}$ are less than unity, and $\kappa'' > \frac{\sqrt{(\kappa''^2-\kappa^2)}}{k} > \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}$. Since, when $\theta = 0$, ϕ becomes $\frac{1}{2}\pi$, we deduce

$$\int_0^\theta \frac{(\alpha+\beta\sin^2\theta)\,d\theta}{\Delta(\kappa, \kappa', \kappa'', \theta)} = \int_\phi^{\frac{1}{2}\pi} \frac{(\alpha'+\beta'\sin^2\phi)\,d\phi}{\Delta\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}} \quad \text{.........................(21)},$$

whence

$$\int_0^{\frac{1}{2}\pi} \frac{(\alpha+\beta\sin^2\theta)\,d\theta}{\Delta(\kappa, \kappa', \kappa'', \theta)} = \int_0^{\frac{1}{2}\pi} \frac{(\alpha'+\beta'\sin^2\phi)\,d\phi}{\Delta\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}} \quad \text{.........................(22)},$$

and

$$L\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\},$$

$$M\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}$$

can be expressed by

$L(\kappa, \kappa', \kappa'', \frac{1}{2}\pi)$, $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \frac{1}{2}\pi)$, $M(\kappa, \kappa', \kappa'', \theta)$.

33. We have also $\frac{kk'\sin^4\theta\,d\theta}{\Delta(\kappa, \kappa', \kappa'', \theta)}$

$$= -\frac{k''^4}{\kappa''^4}\frac{1}{(1-\kappa''^2\sin^2\phi)}\frac{d\phi}{\Delta\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}}$$

$$+\frac{1}{\kappa''^2}\cdot\frac{\sin^2\phi\,d\phi}{\Delta\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}}$$

$$+\frac{1}{\kappa''^2}\left(\frac{1}{\kappa''^2}-2\right)\frac{d\phi}{\Delta\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}} \text{...(23)},$$

so that $P\left\{-\kappa''^2, \dfrac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \dfrac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}$

depends on

$$L(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi),\ L(\kappa, \kappa', \kappa'', \theta),\ M(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi),\ M(\kappa, \kappa', \kappa'', \theta),$$
$$N(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi),\ N(\kappa, \kappa', \kappa'', \theta),$$

and it follows from (20) that $Q(\kappa, \kappa', \kappa'', \theta)$ depends on

$$L(\kappa, \kappa', \kappa'', \theta),\ M(\kappa, \kappa', \kappa'', \theta),\ N(\kappa, \kappa', \kappa'', \theta),$$
$$N\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \tfrac{1}{2}\pi\right\}$$
$$-N\left\{\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k}, \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}, \kappa'', \phi\right\}.$$

34. If X denotes $x(1-x)(1-\kappa^2x)(1-\kappa'^2x)(1-\kappa''^2x)$, since we suppose that $\kappa''>\kappa'>\kappa$, it follows that for all values of x increasing from $\left\{\begin{array}{c} -\infty \text{ to } 0 \\ 1 \text{ to } \dfrac{1}{\kappa''^2} \\ \dfrac{1}{\kappa'^2} \text{ to } \dfrac{1}{\kappa^2} \end{array}\right\}$ X is negative; while for values of x increasing from $\left\{\begin{array}{c} 0 \text{ to } 1 \\ \dfrac{1}{\kappa''^2} \text{ to } \dfrac{1}{\kappa'^2} \\ \dfrac{1}{\kappa^2} \text{ to } +\infty \end{array}\right\}$ X is positive.

Let us put

$$I_1=\int_{-\infty}^{x}\frac{(\alpha+\beta x)\,dx}{\sqrt{(-X)}},\quad I_3=\int_{1}^{x}\frac{(\alpha+\beta x)\,dx}{\sqrt{(-X)}},\quad I_5=\int_{\frac{1}{\kappa'^2}}^{x}\frac{(\alpha+\beta x)\,dx}{\sqrt{(-X)}},$$

$$I_2=\int_{0}^{x}\frac{(\alpha+\beta x)\,dx}{\sqrt{(X)}},\quad I_4=\int_{\frac{1}{\kappa''^2}}^{x}\frac{(\alpha+\beta x)\,dx}{\sqrt{(X)}},\quad I_6=\int_{\frac{1}{\kappa^2}}^{x}\frac{(\alpha+\beta x)\,dx}{\sqrt{(X)}},$$

the limits of these integrals being such that between them X retains the same sign; α and β denote arbitrary quantities. The superior limits of I_1, I_3, I_5 are thus respectively zero, $\frac{1}{\kappa''^2}$, and $\frac{1}{\kappa^2}$; and the superior limits of I_2, I_4, I_6 are respectively 1, $\frac{1}{\kappa'^2}$, and $+\infty$. All these integrals can be put under the form $\int_0^\theta \frac{(l+m\sin^2\theta)\,d\theta}{\sqrt{\{(1-p^2\sin^2\theta)(1-q^2\sin^2\theta)(1-r^2\sin^2\theta)\}}}$, where p, q, and r are less than unity; these quantities depend only on κ, κ', κ''; while l and m contain in addition α and β. As has been already observed I_2 is reduced to this form by putting $x=\sin^2\theta$. The reduction of I_4 to the required form is effected by the assumption $x=\frac{k'^2-(\kappa''^2-\kappa'^2)\sin^2\phi}{\kappa''^2k'^2-(\kappa''^2-\kappa'^2)\sin^2\phi}$, which gives $\phi=0$ for $x=\frac{1}{\kappa''^2}$ and $\phi=\frac{1}{2}\pi$ for $x=\frac{1}{\kappa'^2}$. By means of this transformation, we find

$$I_4=\frac{2}{\sqrt{(\kappa''^2-\kappa^2)}\,k'^3\kappa''}\int_0^\phi \frac{\{k'^2(\alpha\kappa''^2+\beta)-(\kappa''^2-\kappa'^2)(\alpha+\beta)\sin^2\phi\}\,d\phi}{\sqrt{\left[\left(1-\frac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\phi\right)\left(1-\frac{\kappa''^2-\kappa'^2}{\kappa''^2k'^2}\sin^2\phi\right)\left\{1-\frac{k^2(\kappa''^2-\kappa'^2)}{k'^2(\kappa''^2-\kappa^2)}\sin^2\phi\right\}\right]}} \quad \ldots(24),$$

$\frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}$, $\frac{\sqrt{(\kappa''^2-\kappa'^2)}}{\kappa''k'}$, $\frac{k\sqrt{(\kappa''^2-\kappa'^2)}}{k'\sqrt{(\kappa''^2-\kappa^2)}}$ are all <1 and such that

$$\frac{k\sqrt{(\kappa''^2-\kappa'^2)}}{k'\sqrt{(\kappa''^2-\kappa^2)}}>\frac{\sqrt{(\kappa''^2-\kappa'^2)}}{\kappa''k'}>\frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'}.$$

35. Employing the transformation already noticed in section 32, which becomes, in the present case,

$$\frac{k''}{k'}\sqrt{\left(\frac{\kappa'^2-\kappa^2}{\kappa''^2-\kappa^2}\right)}\tan\theta\tan\phi=1,$$

we find for I_4 the following expression also:

$$I_4=\frac{2}{k'\kappa'\sqrt{(\kappa''^2-\kappa^2)}}\int_0^{\frac{1}{2}\pi}\frac{(\alpha'+\beta'\sin^2\theta)\,d\theta}{\sqrt{\left[\left(1-\frac{\kappa''^2-\kappa'^2}{\kappa''^2-\kappa^2}\sin^2\theta\right)\left(1-\frac{\kappa^2(\kappa''^2-\kappa'^2)}{\kappa'^2(\kappa''^2-\kappa^2)}\sin^2\theta\right)\left(1-\frac{k^2(\kappa''^2-\kappa'^2)}{k'^2(\kappa''^2-\kappa^2)}\sin^2\theta\right)\right]}}\quad\ldots(25),$$

where
$$\alpha'=\alpha\kappa'^2+\beta,\quad \beta'=-\frac{(\alpha\kappa^2+\beta)(\kappa''^2-\kappa'^2)}{\kappa''^2-\kappa^2}.$$

36. To reduce I_6 to the required form, we observe that in it x ranges from $\frac{1}{\kappa^2}$ to $+\infty$; we shall therefore put $x=\frac{\operatorname{cosec}^2\phi}{\kappa^2}$, which transformation gives $\phi=0$ for $x=\infty$, and $\phi=\frac{1}{2}\pi$ for $x=\frac{1}{\kappa^2}$. We thus deduce the following:

$$I_6=\frac{2}{\kappa'\kappa''}\int_\phi^{\frac{1}{2}\pi}\frac{(\beta+\alpha\kappa^2\sin^2\phi)\,d\phi}{\sqrt{\left\{(1-\kappa^2\sin^2\phi)\left(1-\frac{\kappa^2}{\kappa'^2}\sin^2\phi\right)\left(1-\frac{\kappa^2}{\kappa''^2}\sin^2\phi\right)\right\}}}\quad\ldots\ldots\ldots\ldots\ldots(26).$$

Since $\frac{\kappa}{\kappa'}>\frac{\kappa}{\kappa''}>\kappa$, we find also, by assuming $\frac{\sqrt{(\kappa''^2-\kappa^2)}}{\kappa'}\tan\theta\tan\phi=1$, and by suitably modifying (21),

$$I_6=\frac{2}{\kappa'^3k\sqrt{(\kappa''^2-\kappa^2)}}\int_0^\theta\frac{\{\kappa'^2(\alpha\kappa^2+\beta)-\kappa^2(\alpha\kappa'^2+\beta)\sin^2\theta\}\,d\theta}{\sqrt{\left[\left(1-\frac{\kappa^2(\kappa''^2-\kappa'^2)}{\kappa'^2(\kappa''^2-\kappa^2)}\sin^2\theta\right)\left(1-\frac{\kappa^2k'^2}{\kappa'^2k^2}\sin^2\theta\right)\left(1-\frac{\kappa^2}{\kappa'^2}\sin^2\theta\right)\right]}}.$$

37. If we assume $x + z = xz$, and put

$$Z = z(1-z)(1-k^2 z)(1-k'^2 z)(1-k''^2 z),$$

we find $X + \dfrac{Z}{(1-z)^6} = 0$, whence $\sqrt{(-X)} = \dfrac{\sqrt{(Z)}}{\pm(1-z)^3}$, and if we regard $\sqrt{(-X)}$ and $\sqrt{(Z)}$ as essentially positive, for values of z which are less than unity $\sqrt{(-X)} = \dfrac{\sqrt{(Z)}}{(1-z)^3}$, and for values of z which exceed unity $\sqrt{(-X)} = \dfrac{\sqrt{(Z)}}{(z-1)^3}$.

If then we express the differential $\dfrac{(\alpha + \beta x)\,dx}{\sqrt{(-X)}}$ in terms of z, we find for values of z, which are less than unity, the formula

$$\frac{(\alpha + \beta x)\,dx}{\sqrt{(-X)}} = \frac{(\alpha + \beta)\,z - \alpha}{\sqrt{(Z)}}\,dz,$$

and for values of z, which exceed unity, the following formula

$$\frac{(\alpha + \beta x)\,dx}{\sqrt{(-X)}} = \frac{\alpha - (\alpha + \beta)\,z}{\sqrt{(Z)}}\,dz.$$

38. To apply these formulæ to the reduction of the integrals I_1, I_3, I_5 to the normal form; namely,

$$\int \frac{(l + m \sin^2\theta)\,d\theta}{\sqrt{\{(1 - p^2 \sin^2\theta)(1 - q^2 \sin^2\theta)(1 - r^2 \sin^2\theta)\}}},$$

we must observe that when $x = \pm\infty$, then $z = 1$; when $x = 0$, then $z = 0$; when $x = 1$, z becomes $\pm\infty$; and when x takes the values $\dfrac{1}{\kappa^2}$, $\dfrac{1}{\kappa'^2}$, $\dfrac{1}{\kappa''^2}$, the corresponding values of z are $\dfrac{1}{k^2}$, $\dfrac{1}{k'^2}$, $\dfrac{1}{k''^2}$.

39. As in I_1 x ranges from $-\infty$ to 0, the corresponding values of z are included between 1 and 0; hence

$$I_1 = \int_1^z \frac{(\alpha + \beta)\,z - \alpha}{\sqrt{(Z)}}\,dz = \int_z^1 \frac{\alpha - (\alpha + \beta)\,z}{\sqrt{(Z)}}\,dz,$$

and putting $z=\sin^2\theta$ or $x=-\tan^2\theta$, we find

$$I_1=2\int_\theta^{\frac{1}{2}\pi}\frac{\{\alpha-(\alpha+\beta)\sin^2\theta\}\,d\theta}{\sqrt{\{(1-k^2\sin^2\theta)(1-k'^2\sin^2\theta)(1-k''^2\sin^2\theta)\}}}\ldots(27),$$

and since $k>k'>k''$ by assuming $\kappa\tan\theta\tan\phi=1$, we find

$$I_1=\frac{2}{\kappa'\kappa''}\int_0^\phi\frac{\{(\alpha\kappa^2+\beta)\sin^2\phi-\beta\}\,d\phi}{\sqrt{\left\{(1-k^2\sin^2\phi)\left(1-\frac{k^2-k'^2}{\kappa'^2}\sin^2\phi\right)\left(1-\frac{k^2-k''^2}{\kappa''^2}\sin^2\phi\right)\right\}}}.$$

40. As in I_8 x ranges from 1 to $\frac{1}{\kappa''^2}$, the corresponding values of z are $+\infty$ and $\frac{1}{k''^2}$; so that the values of z corresponding to the values of x being greater than $\frac{1}{k''^2}$, we have

$$I_8=\int_\infty^z\frac{\alpha-(\alpha+\beta)z}{\sqrt{(Z)}}\,dz,$$

and between the limits Z preserves the sign +.

The value of I_8 is therefore derived from the expression for I_6 (26) by changing the limits of integration in this expression from ϕ and $\frac{1}{2}\pi$ to 0 and ϕ, and by writing in it k, k', k'' respectively in place of κ'', κ', κ, and for α and β, $-\alpha$ and $(\alpha+\beta)$ respectively. We find thus

$$I_8=\frac{2}{kk'}\int_0^\phi\frac{\{\alpha+\beta-\alpha k''^2\sin^2\phi\}\,d\phi}{\sqrt{\left\{(1-k''^2\sin^2\phi)\left(1-\frac{k''^2}{k^2}\sin^2\phi\right)\left(1-\frac{k''^2}{k'^2}\sin^2\phi\right)\right\}}},$$

where $z=\frac{\operatorname{cosec}^2\phi}{k''^2}$, whence $x=\frac{1}{1-k''^2\sin^2\phi}$.

41. As in I_5 x ranges from $\frac{1}{\kappa'^2}$ to $\frac{1}{\kappa^2}$, it is expressed in terms of z by the integral

$$I_5=\int_{\frac{1}{k'^2}}^z\frac{\{\alpha-(\alpha+\beta)z\}\,dz}{\sqrt{(Z)}},$$

the superior limit of z being $\frac{1}{k^2}$, and its value is derived from the value already given for I_4 (25) by the same processes by which we have just determined the value of I_3 from that of I_6. We thus find the following expression for I_5:

$$I_5 = \frac{2}{k'\kappa'(\kappa''^2-\kappa^2)^{\frac{3}{2}}} \int_0^\theta \frac{\{(\alpha\kappa'^2+\beta)(\kappa''^2-\kappa^2)-(\alpha\kappa''^2+\beta)(\kappa'^2-\kappa^2)\sin^2\theta\}\,d\theta}{\sqrt{\left[\left(1-\frac{k^2-k'^2}{k^2-k''^2}\sin^2\theta\right)\left(1-\frac{k''^2(k^2-k'^2)}{k'^2(k^2-k''^2)}\sin^2\theta\right)\left(1-\frac{\kappa''^2(k^2-k'^2)}{\kappa'^2(k^2-k''^2)}\sin^2\theta\right)\right]}},$$

where

$$z = \frac{k^2-k''^2-(k^2-k'^2)\sin^2\theta}{k'^2(k^2-k''^2)-k''^2(k^2-k'^2)\sin^2\theta},$$

from which we find

$$x = \frac{k^2-k''^2-(k^2-k'^2)\sin^2\theta}{(k^2-k''^2)\kappa'^2-(k^2-k'^2)\kappa''^2\sin^2\theta}.$$

The addition of the functions I_1, I_3, I_4, I_5, I_6 is derived from the formulæ already given for the addition of functions of the form I_2, by substituting in these formulæ for κ, κ', κ'' the values of the corresponding quantities in the remaining functions.

CHAPTER IV.

ON THE PERIODICITY OF THE ELLIPTIC INTEGRALS.

42. In this chapter we shall have occasion to refer to the notation and formulæ for the addition of elliptic functions, which are given in Chapter II. If the moduli k, k' of two elliptic functions of the first kind are connected by the relation $k^2 + k'^2 = 1$, such moduli are said to be complementary, and the corresponding complete functions are denoted respectively by the symbols K, K'. Now if in the elliptic function $F(k, \psi)$ we put $\sin\psi = i \tan\omega$, where $i = \sqrt{(-1)}$, it is transformed into $iF(k', \omega)$, so that if in any of the algebraic equivalents of $F(k, \phi) + F(k, \psi) - F(k, \sigma) = 0$ we put $\sin\psi = i \tan\omega$, we deduce an algebraic equivalent to the equation $F(k, \phi) + iF(k', \omega) - F(k, \sigma) = 0$. The relation between ψ and ω can be written under the forms $\cos\psi = \sec\omega$, $\sqrt{(1 - k^2 \sin^2\psi)} = \dfrac{\sqrt{(1 - k'^2 \sin^2\omega)}}{\cos\omega}$, and is obviously reciprocal. Thus it is easily seen that the last mentioned transcendental equation is equivalent to any of the following equations, see Chapter II., section (9):

$$\cos\sigma = \cos\phi \sec\omega - i \sin\phi \tan\omega \sqrt{(1 - k^2 \sin^2\sigma)},$$

$$\cos\phi = \cos\sigma \sec\omega + i \sin\sigma \tan\omega \sqrt{(1 - k^2 \sin^2\phi)},$$

$$\frac{\sqrt{\{(1-k^2 \sin^2\phi)(1-k'^2 \sin^2\omega)(1-k^2 \sin^2\sigma)\}}}{\cos\omega} = k'^2 + k^2 \cos\phi \sec\omega \cos\sigma,$$

which lead to the following:

$$\cos\phi = \cos\sigma \cos\omega + i \sin\phi \sin\omega \sqrt{(1 - k^2 \sin^2\sigma)},$$

$$\cos\sigma = \cos\phi \cos\omega - i \sin\sigma \sin\omega \sqrt{(1 - k^2 \sin^2\phi)},$$

$$\sqrt{\{(1-k^2 \sin^2\phi)(1-k'^2 \sin^2\omega)(1-k^2 \sin^2\sigma)\}} = k'^2 \cos\omega + k^2 \cos\phi \cos\sigma,$$

and all these equations are different modes of writing the same relation between ϕ, ω, and σ. To deduce an algebraic equivalent to the equation

$$F(k,\ \phi) + iK' = F(k,\ \sigma),$$

it is sufficient to put $\omega = \frac{1}{2}\pi$ in any of the above equations; we are thus led to the equations

$$\cot\phi = i\sqrt{(1 - k^2 \sin^2\sigma)},\quad \cot\sigma = -\,i\sqrt{(1 - k^2 \sin^2\phi)},$$

$$\sqrt{\{(1 - k^2 \sin^2\phi)\,(1 - k^2 \sin^2\sigma)\}} = k\cos\phi\,\cos\psi,$$

from which we find

$$k\sin\phi\,\sin\sigma = 1.$$

43. Let σ, σ' be two amplitudes satisfying the equations

$$F(k,\ \phi) + iK' = F(k,\ \sigma),\quad F(k,\ \sigma) + iK' = F(k,\ \sigma'),$$

we have then $\quad F(k,\ \phi) + 2iK' = F(k,\ \sigma'),$

but $\quad k\sin\phi\,\sin\sigma = k\sin\sigma\,\sin\sigma' = 1,$

so that $\sin\phi = \sin\sigma'$; also

$$\cot\phi = i\sqrt{(1 - k^2 \sin^2\sigma)},\quad \cot\sigma' = -\,i\sqrt{(1 - k^2 \sin^2\sigma)},$$

so that $\cos\phi = -\cos\sigma'$; and since

$$\cot\sigma = -\,i\sqrt{(1 - k\sin^2\phi)} = i\sqrt{(1 - k^2 \sin^2\sigma')},$$

there results $\sqrt{(1 - k^2 \sin^2\phi)} = -\sqrt{(1 - k^2 \sin^2\sigma')}$.

44. These equations are written in a most expressive form by the employment of Jacobi's notation, see Chapter II.

If then $F(k,\ \phi) = u$, $\phi = am\,(u)$, $\sigma = am\,(u + iK')$, and the equations connecting ϕ and σ' are written as follows:

$$\sin am\,(u + 2iK') = \sin am\,(u),\quad \cos am\,(u + 2iK') = -\cos am\,(u),$$

$$\Delta am\,(u + 2iK') = -\,\Delta am\,(u),$$

where $\quad \Delta\,(\phi) = \sqrt{(1 - k^2 \sin^2\phi)}.$

Now the formula for the addition of elliptic functions gives

$$\sin am\,(u + 2K) = -\sin am\,(u),\quad \sin am\,(u + 4K) = \sin am\,(u),$$

and if m, m' denote any integers positive or negative

$$\sin am\,(u + 2miK' + 4m'K) = \sin am\,(u);$$

this equation expresses the property of double periodicity belonging to elliptic functions, and teaches us, that if an elliptic function u is increased by any multiple of $2iK'$ as well as by any multiple of $4K$ which are its periods, the value of the sine of the amplitude is unaffected by such increase.

Admitting imaginary values of θ, $\sin\theta$, when real, is larger than unity; premising this remark, I propose to determine the value of $F(k, \theta)$ when $\sin\theta = \frac{1}{k}$; this equation can be regarded as resulting from the following system:

$$k' \tan\theta \tan\phi = 1, \quad \sin\phi = i \tan\omega, \quad \omega = \tfrac{1}{2}\pi.$$

Now if $F(k, \theta)$ is transformed by the first of these equations, it becomes $K - F(k, \phi)$; which value becomes $K - iF(k', \omega)$ in consequence of the equation connecting ϕ and ω; or, finally, $K - iK'$ by taking into account the third equation. Hence $\sin am\,(K \pm iK') = \frac{1}{k}$; the values of $\cos am\,(K - iK')$, $\cos am\,(K + iK')$ having opposite signs; as appears also from the formulæ given above by putting $u = K$.

45. If the elliptic function of the second kind

$$\int \sqrt{(1 - k^2 \sin^2\phi)}\, d\phi, \quad \text{or } E(k, \phi)$$

is transformed by the equation $\sin\phi = i \tan\omega$, it becomes $i \int \frac{\sqrt{(1 - k'^2 \sin^2\omega)}}{\cos^2\omega}\, d\omega$. Now

$$\frac{d}{d\omega} \tan\omega \sqrt{(1 - k'^2 \sin^2\omega)}$$

$$= \frac{\sqrt{(1 - k'^2 \sin^2\omega)}}{\cos^2\omega} + \sqrt{(1 - k'^2 \sin^2\omega)} - \frac{1}{\sqrt{(1 - k'^2 \sin^2\omega)}},$$

so that, by this transformation, $E(k, \phi)$ becomes

$$i\,\{\tan\omega \sqrt{(1 - k'^2 \sin^2\omega)} + F(k', \omega) - E(k', \omega)\} \ldots (28).$$

CHAPTER V.

ON THE PERIODS OF THE FIRST CLASS OF HYPER-ELLIPTIC INTEGRALS.

46. FOR the determination of the periods of these functions I shall employ a mode of investigation precisely similar to that which has been already adopted for the determination of the periods of the elliptic functions.

We must revert then to the notation of Chapter III., and to the equations given in that chapter for the addition of the first class of hyper-elliptic integrals. If we put $\sin\theta = i\tan\omega$, $L(\kappa, \kappa', \kappa'', \theta)$ becomes

$$i\,[L(k, k', k'', \omega) - M(k, k', k'', \omega)],$$

and $M(\kappa, \kappa', \kappa'', \theta)$ is transformed into $-iM(k, k', k'', \omega)$; so that the transcendental system

$$\left.\begin{aligned} L(\kappa,\kappa',\kappa'',\phi)+L(\kappa,\kappa',\kappa'',\psi)+i\,[L(k,k',k'',\omega)-M(k,k',k'',\omega)] \\ -L(\kappa, \kappa', \kappa'', \sigma)-L(\kappa, \kappa', \kappa'', \tau)=0, \\ M(\kappa, \kappa', \kappa'', \phi)+M(\kappa, \kappa', \kappa'', \psi)-iM(k, k', k'', \omega) \\ -M(\kappa, \kappa', \kappa'', \sigma)-M(\kappa, \kappa', \kappa'', \tau)=0, \end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

is equivalent to the equations

$$\left.\begin{aligned} \cos\phi\cos\psi &= \cos\sigma\cos\tau\cos\omega + i\sin\phi\sin\psi\sin\omega\nabla(\sigma, \tau) \\ \cos\sigma\cos\tau &= \cos\phi\cos\psi\cos\omega - i\sin\sigma.\sin\tau\sin\omega\nabla(\phi, \psi) \end{aligned}\right\}\ldots(30).$$

If in the quantity $\nabla(\phi, \chi)$ we put $\sin\chi = i\tan\omega$, we see easily that it is transformed into

$$\frac{\sin\phi\cos\phi\Delta(k, k', k'', \omega) - i\sin\omega\cos\omega\Delta(\kappa, \kappa', \kappa'', \phi)}{\cos\omega(\sin^2\phi\cos^2\omega + \sin^2\omega)},$$

so that from the transcendental system (29) can be deduced the following equations also:

$$\left.\begin{array}{l}\cos\phi = \cos\sigma\,\cos\tau\,\cos\psi\,\cos\omega + \sin\sigma\,\sin\tau\,\sin\psi \\ \quad\times\left\{\dfrac{\sin\phi\,\cos\phi\Delta\,(k,\,k',\,k'',\,\omega) - i\,\sin\omega\,\cos\omega\Delta\,(\kappa,\,\kappa',\,\kappa'',\,\phi)}{\sin^2\phi\,\cos^2\omega + \sin^2\omega}\right\} \\ \cos\psi = \cos\sigma\,\cos\tau\,\cos\phi\,\cos\omega + \sin\sigma\,\sin\tau\,\sin\phi \\ \quad\times\left\{\dfrac{\sin\psi\,\cos\psi\Delta\,(k,\,k',\,k'',\,\omega) - i\,\sin\omega\,\cos\omega\Delta\,(\kappa,\,\kappa',\,\kappa'',\,\psi)}{\sin^2\psi\,\cos^2\omega + \sin^2\omega}\right\}\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(31).$$

47. If we put

$$L\,(k,\,k',\,k'',\,\tfrac{1}{2}\pi) - M(k,\,k',\,k'',\,\tfrac{1}{2}\pi) = \Theta,\quad M(k,\,k',\,k'',\,\tfrac{1}{2}\pi) = \Theta',$$

and if ω becomes $\frac{1}{2}\pi$ in the equations (30), we infer that the transcendental system

$$\left.\begin{array}{r}L\,(\kappa,\kappa',\kappa'',\phi) + L\,(\kappa,\kappa',\kappa'',\psi) + i\Theta - L\,(\kappa,\kappa',\kappa'',\sigma) \\ -L\,(\kappa,\kappa',\kappa'',\tau) = 0 \\ M(\kappa,\kappa',\kappa'',\phi) + M(\kappa,\kappa',\kappa'',\psi) - i\Theta' - M(\kappa,\kappa',\kappa'',\sigma) \\ -M(\kappa,\kappa',\kappa'',\tau) = 0\end{array}\right\}\ldots(32)$$

is equivalent to the equations

$$\cot\phi\,\cot\psi = i\nabla\,(\sigma,\,\tau),\quad \cot\sigma\,\cot\tau = -i\nabla\,(\phi,\,\psi)\ldots(33),$$

which must involve the equation

$$\kappa\kappa'\kappa''\,\sin\phi\,\sin\psi\,\sin\sigma\,\sin\tau = 1\ \ldots\ldots\ldots\ (34),$$

which is obtained by putting $\omega = \frac{1}{2}\pi$ in either of the equations (31).

48. If the quantities σ, τ, σ', τ' are connected by the equations

$$L\,(\kappa,\,\kappa',\,\kappa'',\,\sigma) + L\,(\kappa,\,\kappa',\,\kappa'',\,\tau) + i\Theta\ - L\,(\kappa,\,\kappa',\,\kappa'',\,\sigma')$$
$$-\,L\,(\kappa,\,\kappa',\,\kappa'',\,\tau') = 0,$$
$$M(\kappa,\,\kappa',\,\kappa'',\,\sigma) + M(\kappa,\,\kappa',\,\kappa'',\,\tau) - i\Theta' - M(\kappa,\,\kappa',\,\kappa'',\,\sigma')$$
$$-\,M(\kappa,\,\kappa',\,\kappa'',\,\tau') = 0,$$

we deduce by taking into account (33), (34),

$$\kappa\kappa'\kappa''\,\sin\sigma\,\sin\tau\,\sin\sigma'\,\sin\tau' = 1,\quad \cot\sigma'\,\cot\tau' - i\nabla\,(\sigma,\,\tau);$$

whence we infer that the system

$$\left.\begin{aligned} L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + 2i\Theta - L(\kappa, \kappa', \kappa'', \sigma') \\ - L(\kappa, \kappa', \kappa'', \tau') = 0 \\ M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) - 2i\Theta' - M(\kappa, \kappa', \kappa'', \sigma') \\ - M(\kappa, \kappa', \kappa'', \tau') = 0 \end{aligned}\right\},$$

is equivalent to the following

$$\sin\phi \sin\psi = \sin\sigma' \sin\tau', \quad \cos\phi \cos\psi = - \cos\sigma' \cos\tau' \ldots (35).$$

We have also the relation

$$\nabla(\phi, \psi) = - \nabla(\sigma', \tau').$$

49. If we refer to the value given for the integral I_1 in Chapter III. (27), we see, if it be taken between its extreme limits, that it depends on Θ, and on Θ'; and that

$$\int_{-\infty}^{0} \frac{dx}{\sqrt{(-X)}} = 2\Theta, \quad \int_{-\infty}^{0} \frac{x\,dx}{\sqrt{(-X)}} = -2\Theta'.$$

50. Let $L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) = u,$

$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) = v,$$

then if $\sin\phi \sin\psi = f(u, v),$

we have $f(u, v) = f(u + 2mi\Theta, \; v - 2mi\Theta'),$

m being any integer positive or negative, and if

$$\cos\phi \cos\psi = f_1(u, v),$$

we have $f_1(u, v) = (-1)^m f_1(u + 2mi\Theta, \; v - 2mi\Theta')$.

51. We deduce from the transcendental system (32), and from its algebraic equivalents (33), (34) that the equations

$$\kappa\kappa'\kappa'' \sin\phi \sin\psi' \sin\sigma \sin\tau = 1, \quad \cot\phi \cot\psi' = - i\nabla(\sigma, \tau)$$

can be replaced by

$$\begin{aligned} L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi') - i\Theta - L(\kappa, \kappa', \kappa'', \sigma) \\ - L(\kappa, \kappa', \kappa'', \tau) = 0, \\ M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi') + i\Theta' - M(\kappa, \kappa', \kappa'', \sigma) \\ - M(\kappa, \kappa', \kappa'', \tau) = 0, \end{aligned}$$

so that an amplitude ψ' such that

$$\sin\psi' = \sin\psi, \quad \cos\psi' = -\cos\psi$$

satisfies the system

$$L(\kappa, \kappa', \kappa'', \psi') = L(\kappa, \kappa', \kappa'', \psi) + 2i\Theta,$$

$$M(\kappa, \kappa', \kappa'', \psi') = M(\kappa, \kappa', \kappa'', \psi) - 2i\Theta',$$

and $\sqrt{\{(1-\kappa^2\sin^2\psi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\}}$

$$= -\sqrt{\{(1-\kappa^2\sin^2\psi')(1-\kappa'^2\sin^2\psi')(1-\kappa''^2\sin^2\psi')\}}.$$

52. I now proceed to determine what $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \theta)$ become when $\sin\theta = \frac{1}{\kappa''}$: this equation can be regarded as resulting from the system $k''\tan\theta\tan\phi = 1$, $\sin\phi = i\tan\omega$, $\omega = \frac{1}{2}\pi$. If we apply these successive transformations to the functions of which we are treating, and if we put

$$L(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi) = \Phi, \quad M(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi) = \Phi',$$

$$\Omega = \frac{1}{kk'}\int_0^{\frac{1}{2}\pi} \frac{(1-k''^2\sin^2\omega)\,d\omega}{\sqrt{\left\{\left(1-\frac{k''^2}{k^2}\sin^2\omega\right)\left(1-\frac{k''^2}{k'^2}\sin^2\omega\right)(1-k''^2\sin^2\omega)\right\}}},$$

$$\Omega' = \frac{1}{kk'}\int_0^{\frac{1}{2}\pi} \frac{d\omega}{\sqrt{\left\{\left(1-\frac{k''^2}{k^2}\sin^2\omega\right)\left(1-\frac{k''^2}{k'^2}\sin^2\omega\right)(1-k''^2\sin^2\omega)\right\}}},$$

we find that $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \theta)$ become respectively $\Phi - i\Omega$, $\Phi' - i\Omega'$, when $\sin\theta = \frac{1}{\kappa''}$.

If we refer to the value given for the integral I_3 in Chapter III., section 40, we shall see that its complete value depends on Ω and on Ω', and that

$$\int_1^{\frac{1}{\kappa''^2}} \frac{dx}{\sqrt{(-X)}} = 2\Omega, \quad \int_1^{\frac{1}{\kappa''^2}} \frac{x\,dx}{\sqrt{(-X)}} = 2\Omega'.$$

From what has been observed in section (51) it follows that when $\sin\theta = \frac{1}{\kappa''}$, $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \theta)$ become respectively

$$\Phi - i\Omega + 4mi\Theta,\quad \Phi' - i\Omega' - 4mi\Theta',$$

or $$\Phi - i\Omega + (4m+2)\,i\Theta,\quad \Phi' - i\Omega' - (4m+2)\,i\Theta';$$

where m denotes any integer, positive or negative, the values of $\cos\theta$ corresponding to these systems having opposite signs.

53. Let us now seek the algebraic forms of the transcendental system

$$\left.\begin{array}{l} L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + \Phi - i\Omega \\ \qquad - L(\kappa, \kappa', \kappa'', \sigma) - L(\kappa, \kappa', \kappa'', \tau) = 0 \\ M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) + \Phi' - i\Omega' \\ \qquad - M(\kappa, \kappa', \kappa'', \sigma) - M(\kappa, \kappa', \kappa'', \tau) = 0 \end{array}\right\} \ldots(36),$$

and for this purpose we shall refer to the algebraic system found by interchanging κ and κ'' in (17) Chapter III.; if in this we put $\sin\chi = \frac{1}{\kappa''}$, we deduce the following as an algebraic representation of the system (36):

$$\left.\begin{array}{l} (1 - \kappa''^2 \sin^2\sigma)(1 - \kappa''^2 \sin^2\tau) \\ \quad + \kappa'' \sin\phi \sin\psi\, [k''^2 \nabla'(\sigma, \tau) + \kappa''^2 \cos\sigma \cos\tau \nabla(\sigma, \tau)] = 0 \\ (1 - \kappa''^2 \sin^2\phi)(1 - \kappa''^2 \sin^2\psi) \\ \quad - \kappa'' \sin\sigma \sin\tau\, [k''^2 \nabla'(\phi, \psi) + \kappa''^2 \cos\phi \cos\psi \nabla(\phi, \psi)] = 0 \end{array}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Again, if we interchange κ and κ'' in equation (14) Chapter III., and put $\sin\chi = \frac{1}{\kappa''}$, we find

$$\frac{ik''\kappa'' \sqrt{\{(\kappa''^2 - \kappa^2)(\kappa''^2 - \kappa'^2)\}} \sin\phi \sin\psi \sin\sigma \sin\tau}{\sqrt{\{(1 - \kappa''^2 \sin^2\phi)(1 - \kappa''^2 \sin^2\psi)(1 - \kappa''^2 \sin^2\sigma)(1 - \kappa''^2 \sin^2\tau)\}}} = 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

which is involved in the system (37).

54. If we suppose that σ', τ' satisfy the relations

$$\begin{aligned} &L(\kappa, \kappa', \kappa'', \sigma) + L(\kappa, \kappa', \kappa'', \tau) + \Phi - i\Omega \\ &\qquad - L(\kappa, \kappa', \kappa'', \sigma') - L(\kappa, \kappa', \kappa'', \tau') = 0, \\ &M(\kappa, \kappa', \kappa'', \sigma) + M(\kappa, \kappa', \kappa'', \tau) + \Phi' - i\Omega' \\ &\qquad - M(\kappa, \kappa', \kappa'', \sigma') - M(\kappa, \kappa', \kappa'', \tau') = 0; \end{aligned}$$

we deduce from the system (37)

$$\frac{(1 - \kappa''^2 \sin^2\sigma')(1 - \kappa''^2 \sin^2\tau')}{k''^2 \nabla'(\sigma', \tau') + \kappa''^2 \cos\sigma' \cos\tau' \nabla(\sigma', \tau')} = -\kappa'' \sin\sigma \sin\tau,$$

$$\frac{(1 - \kappa''^2 \sin^2\sigma)(1 - \kappa''^2 \sin^2\tau)}{k''^2 \nabla'(\sigma, \tau) + \kappa''^2 \cos\sigma \cos\tau \nabla(\sigma, \tau)} = \kappa'' \sin\sigma' \sin\tau',$$

so that if

$$\begin{aligned} &L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + 2\Phi - 2i\Omega \\ &\qquad - L(\kappa, \kappa', \kappa'', \sigma') - L(\kappa, \kappa', \kappa'', \tau') = 0, \\ &M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) + 2\Phi' - 2i\Omega' \\ &\qquad - M(\kappa, \kappa', \kappa'', \sigma') - M(\kappa, \kappa', \kappa'', \tau') = 0, \end{aligned}$$

we infer, in consequence of (37),

$$\sin\phi \sin\psi = -\sin\sigma' \sin\tau' \quad \ldots\ldots\ldots\ldots \quad (39),$$

and again from (38)

$$\sqrt{\{(1 - \kappa''^2 \sin^2\phi)(1 - \kappa''^2 \sin^2\psi)\}} = -\sqrt{\{(1 - \kappa''^2 \sin^2\sigma')(1 - \kappa''^2 \sin^2\tau')\}}.$$

55. I proceed now to determine the values of $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \theta)$, when $\sin\theta = \frac{1}{\kappa'}$; and, adopting the method already employed, I shall transform these functions by the successive substitutions

$$k' \tan\theta \tan\phi = 1, \quad \sin\phi = i \tan\omega, \quad \omega = \tfrac{1}{2}\pi.$$

By means of these transformations $L(\kappa, \kappa', \kappa'', \theta)$ becomes

$$\Phi - \frac{i}{kk''} \int_0^{\frac{1}{2}\pi} \frac{(1 - k'^2 \sin^2\omega)\, d\omega}{\sqrt{\left\{\left(1 - \frac{k'^2}{k^2} \sin^2\omega\right)\left(1 - \frac{k'^2}{k''^2} \sin^2\omega\right)(1 - k'^2 \sin^2\omega)\right\}}}.$$

Now if we put $\sin\epsilon = \frac{k''}{k'}$, the integral

$$\int_0^{\frac{1}{2}\pi} \frac{(1-k'^2\sin^2\omega)\,d\omega}{\sqrt{\left\{\left(1-\frac{k'^2}{k^2}\sin^2\omega\right)\left(1-\frac{k'^2}{k''^2}\sin^2\omega\right)(1-k'^2\sin^2\omega)\right\}}}$$

$$= \int_0^{\epsilon} \frac{(1-k'^2\sin^2\omega)\,d\omega}{\sqrt{\left\{\left(1-\frac{k'^2}{k^2}\sin^2\omega\right)\left(1-\frac{k'^2}{k''^2}\sin^2\omega\right)(1-k'^2\sin^2\omega)\right\}}}$$

$$-i\int_{\epsilon}^{\frac{1}{2}\pi} \frac{(1-k'^2\sin^2\omega)\,d\omega}{\sqrt{\left\{\left(1-\frac{k'^2}{k^2}\sin^2\omega\right)\left(\frac{k'^2}{k''^2}\sin^2\omega-1\right)(1-k'^2\sin^2\omega)\right\}}},$$

and if we denote the first of these integrals by U and the second by U', we find that when $\sin\theta = \frac{1}{\kappa'}$, $L(\kappa, \kappa', \kappa'', \theta)$ becomes

$$\Phi - \frac{1}{kk''}(U' + iU).$$

If we put $\sin\omega = \sin\epsilon \sin\psi$, we find that $U = kk''\Omega$; and if we put

$$\sin\omega = \frac{\sin\epsilon}{\sqrt{(1-\cos^2\epsilon\sin^2\chi)}},$$

$$\frac{k'\sqrt{(\kappa''^2-\kappa^2)}}{\kappa''}\,\Xi=\int_0^{\frac{1}{2}\pi}\frac{\left(1-\frac{\kappa''^2-\kappa'^2}{\kappa''^2k'^2}\sin^2\chi\right)d\chi}{\sqrt{\left\{\left(1-\frac{k^2(\kappa''^2-\kappa'^2)}{k'^2(\kappa''^2-\kappa^2)}\sin^2\chi\right)\left(1-\frac{\kappa''^2-\kappa'^2}{\kappa''^2k'^2}\sin^2\chi\right)\left(1-\frac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\chi\right)\right\}}},$$

we find that $U'=kk''\Xi$; so that when $\sin\theta=\frac{1}{\kappa'}$, $L(\kappa,\kappa',\kappa'',\theta)$ becomes $\Phi-\Xi-i\Omega$.

By a method precisely similar, and by putting

$$k'\kappa''\sqrt{(\kappa''^2-\kappa^2)}\,\Xi'=\int_0^{\frac{1}{2}\pi}\frac{\left(1-\frac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\chi\right)d\chi}{\sqrt{\left\{\left(1-\frac{k^2(\kappa''^2-\kappa'^2)}{k'^2(\kappa''^2-\kappa^2)}\sin^2\chi\right)\left(1-\frac{\kappa''^2-\kappa'^2}{\kappa''^2k'^2}\sin^2\chi\right)\left(1-\frac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\chi\right)\right\}}},$$

we find that when $\sin\theta=\frac{1}{\kappa'}$, $M(\kappa,\kappa',\kappa'',\theta)$ becomes $\Phi'-\Xi'-i\Omega'$.

By referring to the value given for the integral I_4 (24) in Chapter III., we find that its complete value is expressed by means of Ξ and Ξ', and that we have

$$\int_{\frac{1}{\kappa''^2}}^{\frac{1}{\kappa'^2}}\frac{dx}{\sqrt{(X)}}=2\Xi,\quad\int_{\frac{1}{\kappa''^2}}^{\frac{1}{\kappa'^2}}\frac{xdx}{\sqrt{(X)}}=2\Xi'.$$

56. By interchanging κ'' and κ' in the system (37), we find that the transcendental system

$$L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + \Phi - \Xi - i\Omega$$
$$- L(\kappa, \kappa', \kappa'', \sigma) - L(\kappa, \kappa', \kappa'', \tau) = 0,$$
$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) + \Phi' - \Xi' - i\Omega'$$
$$- M(\kappa, \kappa', \kappa'', \sigma) - M(\kappa, \kappa', \kappa'', \tau) = 0$$

is equivalent to the algebraic equations

$$(1 - \kappa'^2 \sin^2\sigma)(1 - \kappa'^2 \sin^2\tau)$$
$$+ \kappa' \sin\phi \sin\psi\, [k'^2 \nabla'(\sigma, \tau) + \kappa'^2 \cos\sigma \cos\tau \nabla(\sigma, \tau)] = 0,$$
$$(1 - \kappa'^2 \sin^2\phi)(1 - \kappa'^2 \sin^2\psi)$$
$$- \kappa' \sin\sigma \sin\tau\, [k'^2 \nabla'(\phi, \psi) + \kappa'^2 \cos\phi \cos\psi \nabla(\phi, \psi)] = 0,$$

and we deduce also, by means of (14) Chapter III.,

$$\frac{k'\kappa' \sqrt{\{(\kappa'^2 - \kappa^2)(\kappa''^2 - \kappa'^2)\}} \sin\phi \sin\psi \sin\sigma \sin\tau}{\sqrt{\{(1 - \kappa'^2 \sin^2\phi)(1 - \kappa'^2 \sin^2\psi)(1 - \kappa'^2 \sin^2\sigma)(1 - \kappa'^2 \sin^2\tau)\}}} = 1.$$

We thus find, as in section 54, that if

$$L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + 2\Phi - 2\Xi - 2i\Omega$$
$$- L(\kappa, \kappa', \kappa'', \sigma') - L(\kappa, \kappa', \kappa'', \tau') = 0,$$
$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) + 2\Phi' - 2\Xi' - 2i\Omega'$$
$$- M(\kappa, \kappa', \kappa'', \sigma') - M(\kappa, \kappa', \kappa'', \tau') = 0,$$
$$\sin\phi \sin\psi = - \sin\sigma' \sin\tau',$$
$$\sqrt{\{(1 - \kappa'^2 \sin^2\phi)(1 - \kappa'^2 \sin^2\psi)\}} = - \sqrt{\{(1 - \kappa'^2 \sin^2\sigma')(1 - \kappa'^2 \sin^2\tau')\}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

It also follows that when $\sin\theta = \frac{1}{\kappa'}$, $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \theta)$ become respectively

$$\Phi - \Xi - i\Omega + 4mi\Theta, \quad \Phi' - \Xi' - i\Omega' - 4mi\Theta',$$

or $\Phi - \Xi - i\Omega + (4m + 2)\, i\Theta$, $\Phi' - \Xi' - i\Omega' - (4m + 2)\, i\Theta'$,

where m is any integer, positive or negative, and the values of $\cos\theta$ corresponding to these systems have opposite signs.

57. To determine what $L(\kappa, \kappa', \kappa'', \theta)$ and $M(\kappa, \kappa', \kappa'', \theta)$ become when $\sin\theta = \dfrac{1}{\kappa}$, we shall regard this equation as resulting from the system

$$\sin\theta = i\tan\omega,\quad \kappa\tan\omega\tan\omega' = 1,\quad \sin\omega' = i\tan\omega'',\quad \omega'' = \tfrac{1}{2}\pi;$$

and if we apply these transformations in succession to the functions $L(\kappa, \kappa', \kappa'', \theta)$ and $M(\kappa, \kappa', \kappa'', \theta)$, we find, by putting

$$\kappa'\kappa''\Sigma = \kappa^2\int_0^{\frac{1}{2}\pi} \frac{\sin^2\omega''\,d\omega''}{\sqrt{\left\{(1-\kappa^2\sin^2\omega'')\left(1-\frac{\kappa^2}{\kappa'^2}\sin^2\omega''\right)\left(1-\frac{\kappa^2}{\kappa''^2}\sin^2\omega''\right)\right\}}},$$

$$\kappa'\kappa''\Sigma' = \int_0^{\frac{1}{2}\pi} \frac{d\omega''}{\sqrt{\left\{(1-\kappa^2\sin^2\omega'')\left(1-\frac{\kappa^2}{\kappa'^2}\sin^2\omega''\right)\left(1-\frac{\kappa^2}{\kappa''^2}\sin^2\omega''\right)\right\}}},$$

that $L(\kappa, \kappa', \kappa'', \theta)$ becomes $i\Theta - \Sigma$, and that $M(\kappa, \kappa', \kappa'', \theta)$ becomes at the same time $-i\Theta' - \Sigma'$. The algebraic forms of the transcendental system

$$L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + i\Theta - \Sigma$$
$$- L(\kappa, \kappa', \kappa'', \sigma) - L(\kappa, \kappa', \kappa'', \tau) = 0,$$
$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) - i\Theta' - \Sigma'$$
$$- M(\kappa, \kappa', \kappa'', \sigma) - M(\kappa, \kappa', \kappa'', \tau) = 0,$$

are found by interchanging κ and κ'' in equations (37), and we derive from the system

$$L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + 2i\Theta - 2\Sigma$$
$$- L(\kappa, \kappa', \kappa'', \sigma') - L(\kappa, \kappa', \kappa'', \tau') = 0,$$
$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) - 2i\Theta' - 2\Sigma'$$
$$- M(\kappa, \kappa', \kappa'', \sigma') - M(\kappa, \kappa', \kappa'', \tau') = 0,$$

the equations $\quad \sin\phi\sin\psi = -\sin\sigma'\sin\tau'$,

$$\sqrt{\{(1-\kappa^2\sin^2\phi)(1-\kappa^2\sin^2\psi)\}} = -\sqrt{\{(1-\kappa^2\sin^2\sigma')(1-\kappa^2\sin^2\tau')\}}$$
$$\text{......................}(41).$$

If we refer to the value given for I_6 in (26) Chapter III., we see that its complete value depends on Σ and Σ'; and that

$$\int_{\frac{1}{\kappa^2}}^{\infty} \frac{dx}{\sqrt{(X)}} = 2\Sigma, \quad \int_{\frac{1}{\kappa^2}}^{\infty} \frac{xdx}{\sqrt{(X)}} = 2\Sigma'.$$

As when $\sin\theta = \frac{1}{\kappa}$, there is an ambiguity of sign in the cosine, the values of $L(\kappa, \kappa', \kappa'', \theta)$, $M(\kappa, \kappa', \kappa'', \theta)$ corresponding to such ambiguity can be obtained in the same way as they have been already obtained for the other similar cases.

58. It is easy to see that if

$$L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) + 4\Phi$$
$$- L(\kappa, \kappa', \kappa'', \sigma') - L(\kappa, \kappa', \kappa'', \tau') = 0,$$
$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) + 4\Phi'$$
$$- M(\kappa, \kappa', \kappa'', \sigma') - M(\kappa, \kappa', \kappa'', \tau') = 0,$$

we have $$\sin\phi \sin\psi = \sin\sigma' \sin\tau'.$$

If we put, as before,

$$L(\kappa, \kappa', \kappa'', \phi) + L(\kappa, \kappa', \kappa'', \psi) = u,$$
$$M(\kappa, \kappa', \kappa'', \phi) + M(\kappa, \kappa', \kappa'', \psi) = v,$$
$$\sin\phi \sin\psi = f(u, v),$$

we find, in consequence of the equation just given, and by taking into account (35), (39), (41), that

$$f(u, v) = f\begin{Bmatrix} u + 2mi\Theta + 4m'\Phi + 4m''(\Phi - i\Omega) + 4m'''(i\Theta - \Sigma), \\ v - 2mi\Theta' + 4m'\Phi' + 4m''(\Phi' - i\Omega') - 4m'''(i\Theta' + \Sigma') \end{Bmatrix},$$

m, m', m'', m''' being positive or negative integers. Hence, if μ, μ', μ'', μ''' denote any integers positive or negative

$$f(u, v) = f\begin{Bmatrix} u + 4\mu\Phi + 4\mu'\Sigma + 2\mu''i\Theta + 4\mu'''i\Omega, \\ v + 4\mu\Phi' + 4\mu'\Sigma' - 2\mu''i\Theta' + 4\mu'''i\Omega' \end{Bmatrix},$$

so that $f(u, v)$ is a periodic function of the two variables

u and v, each of which has four periods, two real and two imaginary, and the nature of the periodicity is indicated by the above equation.

59. It might be thought that the functions Ξ, Ξ' would give rise to another period, but these are connected with the periods Φ, Σ, and Φ', Σ' in a way which I now proceed to point out.

If we refer to the equations (33), we see that they are satisfied by the following values:

$$\phi = \tfrac{1}{2}\pi,\quad \psi = \sin^{-1}\frac{1}{\kappa'},\quad \sigma = \sin^{-1}\frac{1}{\kappa},\quad \tau = \sin^{-1}\frac{1}{\kappa''},$$

which must also satisfy the transcendental system (32) of which they are the algebraical translation. Hence

$$L(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi) + L\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa'}\right) + i\Theta$$
$$- L\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa}\right) - L\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa''}\right) = 0,$$

$$M(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi) + M\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa'}\right) - i\Theta'$$
$$- M\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa}\right) - M\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa''}\right) = 0,$$

and if we substitute in this system the values already found for the quantities which enter into it, it reduces itself to the following:

$$\Phi - \Xi + \Sigma = 0,\quad \Phi' - \Xi' + \Sigma' = 0 \;\ldots\ldots\ldots\; (42),$$

which express relations between certain definite integrals, which depend on the quantities κ, κ', κ''.

60. Jacobi denotes by u_1, u_2, u_3, u_4, u_5, u_6, the complete values of the integrals I_1, I_2, I_3, I_4, I_5, I_6 as defined in section 34; which he proves are connected by the relations

$$u_2 - u_4 + u_6 = 0,\quad u_1 - u_3 + u_5 = 0.$$

The first of these relations is evidently equivalent to the equations (42), which I have just demonstrated; the second results from the first by writing in it k, k', k'' respectively for κ'', κ', κ.

61. The equations (42) can also be demonstrated by a different arrangement. If we refer to the equations (33), (34), we see that the following transcendental system holds:

$$L\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa}\right) + L\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa''}\right) + i\Theta$$

$$- L\left(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi\right) - L\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa'}\right) = 0,$$

$$M\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa}\right) + M\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa''}\right) - i\Theta'$$

$$- M\left(\kappa, \kappa', \kappa'', \tfrac{1}{2}\pi\right) - M\left(\kappa, \kappa', \kappa'', \sin^{-1}\frac{1}{\kappa'}\right) = 0,$$

which, by substituting the values already given for the quantities which it contains, is reduced to

$$2mi\Theta - \Sigma + \Xi - \Phi = 0, \quad -2mi\Theta' - \Sigma' + \Xi' - \Phi' = 0,$$

m being an integer, which must in this case be zero.

62. The real periods of

$$\int_0^\theta \frac{(\alpha + \beta \sin^2\theta)\, d\theta}{\sqrt{\{(1 - \kappa^2 \sin^2\theta)(1 - \kappa'^2 \sin^2\theta)(1 - \kappa''^2 \sin^2\theta)\}}}$$

when multiplied by i are expressed by the imaginary periods of

$$\int_0^\theta \frac{(\alpha + \beta \sin^2\theta)\, d\theta}{\sqrt{\{(1 - k^2 \sin^2\theta)(1 - k'^2 \sin^2\theta)(1 - k''^2 \sin^2\theta)\}}},$$

and *vice versâ*.

CHAPTER VI.

CASES WHERE THE FIRST CLASS OF HYPER-ELLIPTIC INTEGRALS DEPENDS ON ELLIPTIC FUNCTIONS.

63. THERE are cases in which, if certain relations exist between the quantities κ, κ', κ'', the value of

$$\int\frac{R(\sin\theta)\,d\theta}{\sqrt{\{(1-\kappa^2\sin^2\theta)(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)\}}}$$

depends on elliptic functions and on functions of a lower order, circular and logarithmic. $R(\sin\theta)$ denotes a rational function of $\sin\theta$. Supposing, as we have already done, that $\kappa''>\kappa'>\kappa$, if $\kappa=\kappa'\kappa''$, then $L(\kappa', \kappa'', \kappa'\kappa'', \theta)$ depends on elliptic functions of the first kind, as I now proceed to demonstrate. If we put $\sqrt{(\kappa'\kappa'')}\sin\theta=t$, then

$$\frac{d\theta}{\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$

$$=\frac{1}{\sqrt{(\kappa'\kappa'')}}\,\frac{dt}{\sqrt{(1-pt^2+qt^4-pt^6+t^8)}},$$

where $p=\dfrac{(1+\kappa'^2)(1+\kappa''^2)}{\kappa'\kappa''}$, $q=\dfrac{2\kappa'^2\kappa''^2+(\kappa'^2+\kappa''^2)(1+\kappa'^2\kappa''^2)}{\kappa'^2\kappa''^2}$,

and the differential at the right-hand side of this equation can be written

$$\frac{1}{\sqrt{(\kappa'\kappa'')}}\,\frac{dt}{t^2\sqrt{\left\{\left(t^4+\frac{1}{t^4}\right)-p\left(t^2+\frac{1}{t^2}\right)+q\right\}}}.$$

Let $t+\dfrac{1}{t}=u$, and because the origin of

$$\int_0^\theta\frac{d\theta}{\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$

corresponds to $u = \infty$, we must have

$$2t = u - \sqrt{(u^2 - 4)},\quad \frac{2}{t} = u + \sqrt{(u^2 - 4)},$$

and $\displaystyle\int_0^t \frac{dt}{\sqrt{(1 - pt^2 + qt^4 - pt^6 + t^8)}}$

$$= \frac{1}{2}\int_u^\infty \frac{1}{\sqrt{\{u^4 - (p+4)u^2 + 2 + 2p + q\}}} \left\{\frac{udu}{\sqrt{(u^2-4)}} + du\right\},$$

or, if we put $u^2 - 4 = v^2$,

$$\int_0^t \frac{dt}{\sqrt{(1-pt^2+qt^4-pt^6+t^8)}} = \frac{1}{2}\left[\int_u^\infty \frac{du}{\sqrt{\{u^4-(p+4)u^2+2+2p+q\}}} + \int_v^\infty \frac{dv}{\sqrt{\{v^4-(p-4)v^2+2-2p+q\}}}\right].$$

Now

$$u^4 - (p+4)u^2 + 2 + 2p + q = \left\{u^2 - \frac{(1+\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{u^2 - \frac{(\kappa'+\kappa'')^2}{\kappa'\kappa''}\right\},$$

$$v^4 - (p-4)v^2 + 2 - 2p + q = \left\{v^2 - \frac{(1-\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{v^2 - \frac{(\kappa'-\kappa'')^2}{\kappa'\kappa''}\right\},$$

and if we put

$$\frac{1}{u} = \frac{\sqrt{(\kappa'\kappa'')}}{1+\kappa'\kappa''}\sin\theta_1,\quad \frac{1}{v} = \frac{\sqrt{(\kappa'\kappa'')}}{1-\kappa'\kappa''}\sin\theta_2,$$

$$m_1 = \frac{\kappa'+\kappa''}{1+\kappa'\kappa''},\quad m_2 = \frac{\kappa''-\kappa'}{1-\kappa'\kappa''},$$

we have $\displaystyle\frac{dt}{\sqrt{(1-pt^2+qt^4-pt^6+t^8)}}$

$$= \frac{\sqrt{(\kappa'\kappa'')}}{2}\left\{\frac{1}{1+\kappa'\kappa''}\frac{d\theta_1}{\sqrt{(1-m_1^2\sin^2\theta_1)}} + \frac{1}{1-\kappa'\kappa''}\frac{d\theta_2}{\sqrt{(1-m_2^2\sin^2\theta_2)}}\right\}.$$

Hence, because θ, t, θ_1, θ_2 all commence together, we find by integrating

$$L(\kappa', \kappa'', \kappa'\kappa'', \theta) = \frac{1}{2}\left\{\frac{1}{1+\kappa'\kappa''}F(m_1, \theta_1) + \frac{1}{1-\kappa'\kappa''}F(m_2, \theta_2)\right\}.$$

........................(43).

64. In a similar manner we find

$$\frac{\sin^2\theta d\theta}{\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$
$$=\frac{1}{(\kappa'\kappa'')^{\frac{3}{2}}}\frac{t^2dt}{\sqrt{(1-pt^2+qt^4-pt^6+t^8)}},$$

and

$$\int_0^t\frac{t^2dt}{\sqrt{(1-pt^2+qt^4-pt^6+t^8)}}=\frac{1}{2}\left[\int_v^\infty\frac{dv}{\sqrt{\{v^4-(p-4)v^2+2-2p+q\}}}\right.$$
$$\left.-\int_u^\infty\frac{du}{\sqrt{\{u^4-(p+4)u^2+2+2p+q\}}}\right],$$

and introducing as before the amplitudes θ_1, θ_2, we find

$$M(\kappa',\kappa'',\kappa'\kappa'',\theta)=\frac{1}{2\kappa'\kappa''}\left\{\frac{1}{1-\kappa'\kappa''}F(m_2,\theta_2)-\frac{1}{1+\kappa'\kappa''}F(m_1,\theta_1)\right\}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots(44).$$

Hence $L(\kappa', \kappa'', \kappa'\kappa'', \theta)$, $M(\kappa', \kappa'', \kappa'\kappa'', \theta)$ depend on the elliptic functions $F(m_1, \theta_1)$, $F(m_2, \theta_2)$, whose amplitudes are connected with the amplitude θ of the hyper-elliptic integrals by the equations

$$\sin\theta_1=\frac{(1+\kappa'\kappa'')\sin\theta}{1+\kappa'\kappa''\sin^2\theta},\quad \sin\theta_2=\frac{(1-\kappa'\kappa'')\sin\theta}{1-\kappa'\kappa''\sin^2\theta}\dots(45).$$

It appears from these relations that θ, θ_1, θ_2 vanish together, and also attain the value $\frac{1}{2}\pi$ together.

65. We have also the following forms of the equations connecting θ_1 and θ_2 with θ:

$$\left.\begin{aligned}
\cos\theta_1&=\frac{\cos\theta\sqrt{(1-\kappa'^2\kappa''^2\sin^2\theta)}}{1+\kappa'\kappa''\sin^2\theta},\\
\sqrt{(1-m_1^2\sin^2\theta_1)}&=\frac{\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)\}}}{1+\kappa'\kappa''\sin^2\theta}\\
\cos\theta_2&=\frac{\cos\theta\sqrt{(1-\kappa'^2\kappa''^2\sin^2\theta)}}{1-\kappa'\kappa''\sin^2\theta},\\
\sqrt{(1-m_2^2\sin^2\theta_2)}&=\frac{\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)\}}}{1-\kappa'\kappa''\sin^2\theta}
\end{aligned}\right\}$$
$$\dots\dots\dots\dots\dots\dots(46),$$

which lead to $\frac{\tan\theta_1}{\tan\theta_2} = \frac{1+\kappa'\kappa''}{1-\kappa'\kappa''}$, an equation connecting directly θ_1 and θ_2.

66. In consequence of (43) and (44) we see that the transcendental system

$$\left.\begin{aligned} &L(\kappa', \kappa'', \kappa'\kappa'', \phi) + L(\kappa', \kappa'', \kappa'\kappa'', \psi) + L(\kappa', \kappa'', \kappa'\kappa'', \chi) \\ &\qquad - L(\kappa', \kappa'', \kappa'\kappa'', \sigma) - L(\kappa', \kappa'', \kappa'\kappa'', \tau) = 0 \\ &M(\kappa', \kappa'', \kappa'\kappa'', \phi) + M(\kappa', \kappa'', \kappa'\kappa'', \psi) + M(\kappa', \kappa'', \kappa'\kappa'', \chi) \\ &\qquad - M(\kappa', \kappa'', \kappa'\kappa'', \sigma) - M(\kappa', \kappa'', \kappa'\kappa'', \tau) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(47)$$

can be transformed into the following:

$$\left.\begin{aligned} F(m_1, \phi_1) + F(m_1, \psi_1) + F(m_1, \chi_1) - F(m_1, \sigma_1) - F(m_1, \tau_1) = 0 \\ F(m_2, \phi_2) + F(m_2, \psi_2) + F(m_2, \chi_2) - F(m_2, \sigma_2) - F(m_2, \tau_2) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(48),$$

where the amplitudes ϕ_1, ϕ_2 are derived from ϕ in the same way as θ_1 and θ_2 in the previous section are derived from θ, and so for the others. Hence the addition of functions of the form $\alpha L(\kappa', \kappa'', \kappa'\kappa'', \theta) + \beta M(\kappa', \kappa'', \kappa'\kappa'', \theta)$, where α and β are arbitrary quantities, depends on the addition of functions of the form $F(m_1, \theta_1)$ and on the addition of functions of the form $F(m_2, \theta_2)$, and the formulæ requisite for the solution of this problem are given in Chapter II. Each equation of the system (48) has, independently of the other, an algebraic equivalent, expressing an algebraic relation between the trigonometric functions of the amplitudes $\phi_1, \psi_1, \chi_1, \sigma_1, \tau_1$ for the first, and a similar relation between the trigonometric functions of $\phi_2, \psi_2, \chi_2, \sigma_2, \tau_2$ for the second; if in each of these we pass from the amplitudes ϕ_1, ψ_1, &c., ϕ_2, ψ_2, &c. to ϕ, ψ, &c. we arrive at the algebraic equivalents of the system (47) in which the amplitudes $\phi, \psi, \chi, \sigma, \tau$ alone appear.

67. There exists an elegant equation connecting ϕ and ψ with the amplitudes $\phi_1, \psi_1, \phi_2, \psi_2$ which I now proceed to investigate.

Let
$$F(m_1,\phi_1)+F(m_1,\psi_1)=F(m_1,\zeta_1),$$
$$F(m_2,\phi_2)+F(m_2,\psi_2)=F(m_2,\zeta_2),$$
then, in consequence of the third equation given in section 8, we have

$$\cot\zeta_1=\frac{\sin\phi_1\cos\phi_1\sqrt{(1-m_1^2\sin^2\psi_1)}-\sin\psi_1\cos\psi_1\sqrt{(1-m_1^2\sin^2\phi_1)}}{\sin^2\phi_1-\sin^2\psi_1},$$

$$\cot\zeta_2=\frac{\sin\phi_2\cos\phi_2\sqrt{(1-m_2^2\sin^2\psi_2)}-\sin\psi_2\cos\psi_2\sqrt{(1-m_2^2\sin^2\phi_2)}}{\sin^2\phi_2-\sin^2\psi_2}.$$

Now we find, from the equations (45), (46),

$$\sin\phi_1\sin\phi_2=\frac{(1-\kappa'^2\kappa''^2)\sin^2\phi}{1-\kappa'^2\kappa''^2\sin^4\phi},$$

$$\cos\phi_1\cos\phi_2=\frac{\cos^2\phi\,(1-\kappa'^2\kappa''^2\sin^2\phi)}{1-\kappa'^2\kappa''^2\sin^4\phi},$$

$$\sqrt{\{(1-m_1^2\sin^2\phi_1)(1-m_2^2\sin^2\phi_2)\}}=\frac{(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)}{1-\kappa'^2\kappa''^2\sin^4\phi},$$

$$\sin\phi_1\cos\phi_1=\frac{(1+\kappa'\kappa'')\sin\phi\cos\phi\sqrt{(1-\kappa'^2\kappa''^2\sin^2\phi)}}{(1+\kappa'\kappa''\sin^2\phi)^2},$$

$$\sin\phi_2\cos\phi_2=\frac{(1-\kappa'\kappa'')\sin\phi\cos\phi\sqrt{(1-\kappa'^2\kappa''^2\sin^2\phi)}}{(1-\kappa'\kappa''\sin^2\phi)^2},$$

$$\sqrt{\{(1-m_1^2\sin^2\phi_1)(1-m_2^2\sin^2\psi_2)\}}$$
$$=\frac{\sqrt{\{(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\}}}{(1+\kappa'\kappa''\sin^2\phi)(1-\kappa'\kappa''\sin^2\psi)};$$

whence we deduce, if we put

$$D=\frac{1-\kappa'^2\kappa''^2}{\{(1-\kappa'^2\kappa''^2\sin^4\phi)(1-\kappa'^2\kappa''^2\sin^4\psi)\}^2},$$

$$\{\sin\phi_1\cos\phi_1\sqrt{(1-m_1^2\sin^2\psi_1)}-\sin\psi_1\cos\psi_1\sqrt{(1-m_1^2\sin^2\phi_1)}\}$$
$$\times\{\sin\phi_2\cos\phi_2\sqrt{(1-m_2^2\sin^2\psi_2)}-\sin\psi_2\cos\psi_2\sqrt{(1-m_2^2\sin^2\phi_2)}\}$$
$$=D\left\{\begin{array}{r}\sin^2\phi\,\cos^2\phi\,(1-\kappa'^2\kappa''^2\sin^2\phi)(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\\ (1-\kappa'^2\kappa''^2\sin^4\psi)\\ +\sin^2\psi\,\cos^2\psi\,(1-\kappa'^2\kappa''^2\sin^2\psi)(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)\\ (1-\kappa'^2\kappa''^2\sin^4\phi)\\ -2(1-\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi)\sin\phi\cos\phi\sin\psi\cos\psi\,\Delta(\phi)\,\Delta(\psi)\end{array}\right\},$$

where $\Delta(\phi)$ stands for

$$\sqrt{\{(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)(1-\kappa'^2\kappa''^2\sin^2\phi)\}}.$$

But $$(1-\kappa'^2\kappa''^2\sin^2\phi)(1-\kappa'^2\kappa''^2\sin^4\psi)$$
$$=(1-\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi)(1-\kappa'^2\kappa''^2\sin^2\psi)$$
$$-\kappa'^2\kappa''^2\cos^2\psi(\sin^2\phi-\sin^2\psi),$$
so that this last expression becomes
$$D\left\{\begin{array}{l}(1-\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi)\{\sin\phi\cos\phi\Delta(\psi)-\sin\psi\cos\psi\Delta(\phi)\}^2\\ -\kappa'^2\kappa''^2\cos^2\phi\cos^2\psi(\sin^2\phi-\sin^2\psi)\\ \qquad\qquad\times\{\sin^2\phi(1-\kappa'^2\sin^2\psi)(1-\kappa''^2\sin^2\psi)\\ \qquad\qquad-\sin^2\psi(1-\kappa'^2\sin^2\phi)(1-\kappa''^2\sin^2\phi)\}\end{array}\right\},$$
and also $$(\sin^2\phi_1-\sin^2\psi_1)(\sin^2\phi_2-\sin^2\psi_2)$$
$$=\frac{\{(1-\kappa'^2\kappa''^2)(\sin^2\phi-\sin^2\psi)(1-\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi)\}^2}{\{(1-\kappa'\kappa''^2\sin^4\phi)(1-\kappa'^2\kappa''^2\sin^4\psi)\}^2},$$
so that, by restoring for D its value and remembering our ordinary notation, we find that
$$\cot\zeta_1\cot\zeta_2=\frac{[\nabla(\phi,\psi)]^2-\kappa'^2\kappa''^2\cos^2\phi\cos^2\psi}{(1-\kappa'^2\kappa''^2)(1-\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi)}\quad\ldots(49).$$

68. Suppose now that in the system (47) $\chi=\frac{1}{2}\pi$, we have then $\chi_1=\chi_2=\frac{1}{2}\pi$, and if we put
$$F(m_1,\sigma_1)+F(m_1,\tau_1)=F(m_1,\zeta_1'),$$
$$F(m_2,\sigma_2)+F(m_2,\tau_2)=F(m_2,\zeta_2'),$$
the system (48) becomes in this case
$$F(m_1,\zeta_1)+F(m_1,\tfrac{1}{2}\pi)=F(m_1,\zeta_1'),$$
$$F(m_2,\zeta_2)+F(m_2,\tfrac{1}{2}\pi)=F(m_2,\zeta_2'),$$
whence, by the formula for the addition of elliptic functions given in section 13, we find
$$\tan\zeta_1\tan\zeta_1'=-\frac{1}{\sqrt{(1-m_1^2)}}=-\frac{1+\kappa'\kappa''}{\sqrt{\{(1-\kappa'^2)(1-\kappa''^2)\}}},$$
$$\tan\zeta_2\tan\zeta_2'=-\frac{1}{\sqrt{(1-m_2^2)}}=-\frac{1-\kappa'\kappa''}{\sqrt{\{(1-\kappa'^2)(1-\kappa''^2)\}}},$$
so that, if in the system (47) we put $\chi=\frac{1}{2}\pi$, we have, in consequence of (49),
$$\left\{\frac{[\nabla(\phi,\psi)]^2-\kappa'^2\kappa''^2\cos^2\phi\cos^2\psi}{1-\kappa'^2\kappa''^2\sin^2\phi\sin^2\psi}\right\}\left\{\frac{[\nabla(\sigma,\tau)]^2-\kappa'^2\kappa''^2\cos^2\sigma\cos^2\tau}{1-\kappa'^2\kappa''^2\sin^2\sigma\sin^2\tau}\right\}$$
$$=(1-\kappa'^2)(1-\kappa''^2)(1-\kappa'^2\kappa''^2).$$

This equation, which we have deduced from the formulæ for the addition of elliptic functions, can be derived from the algebraic forms of the hyper-elliptic system (8), (9). If, in the equations (8), (9), we put $\chi = \frac{1}{2}\pi$, we find

$$\nabla(\phi, \psi) = \frac{\cos\phi \cos\psi}{\sin\sigma \sin\tau}, \quad \nabla(\sigma, \tau) = -\frac{\cos\sigma \cos\tau}{\sin\phi \sin\psi},$$

so that

$$\left\{\frac{[\nabla(\phi, \psi)]^2 - \kappa'^2\kappa''^2 \cos^2\phi \cos^2\psi}{1 - \kappa'^2\kappa''^2 \sin^2\phi \sin^2\psi}\right\} \left\{\frac{[\nabla(\sigma, \tau)]^2 - \kappa'^2\kappa''^2 \cos^2\sigma \cos^2\tau}{1 - \kappa'^2\kappa''^2 \sin^2\sigma \sin^2\tau}\right\}$$

$$= \cot^2\phi \cot^2\psi \cot^2\sigma \cot^2\tau,$$

but (11) the right-hand side of the equation is equal to

$$(1 - \kappa'^2)(1 - \kappa''^2)(1 - \kappa'^2\kappa''^2).$$

69. If we refer to the notation and transformations given in section (63) we deduce

$$\frac{\sin^4\theta d\theta}{\sqrt{\{(1 - \kappa'^2 \sin^2\theta)(1 - \kappa''^2 \sin^2\theta)(1 - \kappa'^2\kappa''^2 \sin^2\theta)\}}}$$

$$= \frac{1}{(\kappa'\kappa'')^{\frac{5}{2}}} \frac{t^4 dt}{\sqrt{(1 - pt^2 + qt^4 - pt^6 + t^8)}},$$

and the differential at the right-hand side of this equation is equal to

$$\frac{1}{2(\kappa'\kappa'')^{\frac{5}{2}}} \left[\frac{(u^2 - 1)\, du}{\sqrt{\{u^4 - (p + 4)\, u^2 + 2 + 2p + q\}}}\right.$$

$$\left. - \frac{(v^2 + 1)\, dv}{\sqrt{\{v^4 - (p - 4)\, v^2 + 2 - 2p + q\}}}\right].$$

Now we find easily

$$\int_u^\infty \frac{u^2 du}{\sqrt{\{u^4 - (p + 4)\, u^2 + 2 + 2p + q\}}}$$

$$= \frac{1 + \kappa'\kappa''}{\sqrt{(\kappa'\kappa'')}} \{F(m_1, \theta_1) - E(m_1, \theta_1) - \cot\theta_1 \sqrt{(1 - m_1^2 \sin^2\theta_1)}\},$$

$$\int_v^\infty \frac{v^2 dv}{\sqrt{\{v^4 - (p - 4)\, v^2 + 2 - 2p + q\}}}$$

$$= \frac{1 - \kappa'\kappa''}{\sqrt{(\kappa'\kappa'')}} \{F(m_2, \theta_2) - E(m_2, \theta_2) - \cot\theta_2 \sqrt{(1 - m_2^2 \sin^2\theta_2)}\},$$

so that $\int_0^\theta \frac{\sin^4\theta d\theta}{\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$

$$=\frac{1}{2(\kappa'\kappa'')^{\frac{3}{2}}}\left[\int_v^\infty \frac{(v^2+1)\,dv}{\sqrt{\{v^4-(p-4)v^2+2-2p+q\}}}\right.$$

$$\left.-\int_u^\infty \frac{(u^2-1)\,du}{\sqrt{\{u^4-(p+4)u^2+2+2p+q\}}}\right]$$

$$=\frac{1}{2(\kappa'\kappa'')^3}\left\{\begin{array}{r}(1+\kappa'\kappa'')\{\cot\theta_1\sqrt{(1-m_1^2\sin^2\theta_1)}+E(m_1,\theta_1)\}\\ -\frac{1+\kappa'\kappa''+\kappa'^2\kappa''^2}{1+\kappa'\kappa''}F(m_1,\theta_1)\\ -(1-\kappa'\kappa'')\{\cot\theta_2\sqrt{(1-m_2^2\sin^2\theta_2)}+E(m_2,\theta_2)\}\\ +\frac{1-\kappa'\kappa''+\kappa'^2\kappa''^2}{1-\kappa'\kappa''}F(m_2,\theta_2)\end{array}\right\}$$

$$\dots\dots\dots\dots\dots\dots(50),$$

which, if we adopt the notation of Chapter III., expresses the value of the hyper-elliptic integral $N(\kappa',\kappa'',\kappa'\kappa'',\theta)$ by elliptic functions.

70. We find also in the same way

$$\frac{d\theta}{\cos^2\theta\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$

$$=\frac{\sqrt{(\kappa'\kappa'')}}{\kappa'\kappa''-t^2}\,\frac{dt}{\sqrt{(1-pt^2+qt^4-pt^6+t^8)}},$$

and the differential at the right-hand side of this equation may be written

$$\frac{\sqrt{(\kappa'\kappa'')}}{4}\,\frac{u^2+u\sqrt{(u^2-4)}-2(1+\kappa'\kappa'')}{\{(1+\kappa'\kappa'')^2-\kappa'\kappa''u^2\}}\left\{du+\frac{udu}{\sqrt{(u^2-4)}}\right\}$$

$$\times\frac{1}{\sqrt{\{u^4-(p+4)u^2+2+2p+q\}}},$$

which can be put under the form

$$\frac{\sqrt{(\kappa'\kappa'')}}{2}\cdot\frac{u^2-1-\kappa'\kappa''}{(1+\kappa'\kappa'')^2-\kappa'\kappa''u^2}$$

$$\times\frac{du}{\sqrt{\left[\left\{u^2-\frac{(1+\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{u^2-\frac{(\kappa'+\kappa'')^2}{\kappa'\kappa''}\right\}\right]}}$$

$$+\frac{\sqrt{(\kappa'\kappa'')}}{2}\cdot\frac{v^2+1-\kappa'\kappa''}{(1-\kappa'\kappa'')^2-\kappa'\kappa''v^2}\times\frac{dv}{\sqrt{\left[\left\{v^2-\frac{(1-\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{v^2-\frac{(\kappa'-\kappa'')^2}{\kappa'\kappa''}\right\}\right]}},$$

But $\dfrac{u^2-1-\kappa'\kappa''}{(1+\kappa'\kappa'')^2-\kappa'\kappa''u^2}=\dfrac{1}{\kappa'\kappa''}\left\{\dfrac{1+\kappa'\kappa''}{(1+\kappa'\kappa'')^2-\kappa'\kappa''u^2}-1\right\},$

$$\frac{v^2+1-\kappa'\kappa''}{(1-\kappa'\kappa'')^2-\kappa'\kappa''v^2}=\frac{1}{\kappa'\kappa''}\left\{\frac{1-\kappa'\kappa''}{(1-\kappa'\kappa'')^2-\kappa'\kappa''v^2}-1\right\},$$

so that we have

$$\int_0^\theta\frac{d\theta}{\cos^2\theta\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$

$$=\frac{1}{2\sqrt{(\kappa'\kappa'')}}\left\{\int_u^\infty\left\{1-\frac{1+\kappa'\kappa''}{(1+\kappa'\kappa'')^2-\kappa'\kappa''u^2}\right\}\times\frac{du}{\sqrt{\left[\left\{u^2-\frac{(1+\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{u^2-\frac{(\kappa'+\kappa'')^2}{\kappa'\kappa''}\right\}\right]}}\right\}$$

$$+\frac{1}{2\sqrt{(\kappa'\kappa'')}}\left\{\int_v^\infty\left\{1-\frac{1-\kappa'\kappa''}{(1-\kappa'\kappa'')^2-\kappa'\kappa''v^2}\right\}\times\frac{dv}{\sqrt{\left[\left\{v^2-\frac{(1-\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{v^2-\frac{(\kappa'-\kappa'')^2}{\kappa'\kappa''}\right\}\right]}}\right\}.$$

But by introducing the amplitudes θ_1 and θ_2, and integrating we have

$$\int_u^\infty\frac{1}{\kappa'\kappa''u^2-(1+\kappa'\kappa'')^2}\frac{du}{\sqrt{\left[\left\{u^2-\frac{(1+\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{u^2-\frac{(\kappa'+\kappa'')^2}{\kappa'\kappa''}\right\}\right]}}$$

$$=\frac{\sqrt{(\kappa'\kappa'')}}{(1+\kappa'\kappa'')(1-\kappa'^2)(1-\kappa''^2)}\{\tan\theta_1\sqrt{(1-m_1^2\sin^2\theta_1)}-E(m_1,\theta_1)\},$$

$$\int_v^\infty\frac{1}{\kappa'\kappa''v^2-(1-\kappa'\kappa'')^2}\frac{dv}{\sqrt{\left[\left\{v^2-\frac{(1-\kappa'\kappa'')^2}{\kappa'\kappa''}\right\}\left\{v^2-\frac{(\kappa'-\kappa'')^2}{\kappa'\kappa''}\right\}\right]}}$$

$$=\frac{\sqrt{(\kappa'\kappa'')}}{(1-\kappa'\kappa'')(1-\kappa'^2)(1-\kappa''^2)}\{\tan\theta_2\sqrt{(1-m_2^2\sin^2\theta_2)}-E(m_2,\theta_2)\},$$

so that finally

$$\int_0^\theta \frac{d\theta}{\cos^2\theta\,\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$

$$=\frac{1}{2(1-\kappa'^2)(1-\kappa''^2)}\{\tan\theta_1\sqrt{(1-m_1^2\sin^2\theta_1)}-E(m_1,\theta_1)$$

$$+\tan\theta_2\sqrt{(1-m_2^2\sin^2\theta_2)}-E(m_2,\theta_2)\}$$

$$+\frac{1}{2}\left\{\frac{1}{1+\kappa'\kappa''}F(m_1,\theta_1)+\frac{1}{1-\kappa'\kappa''}F(m_2,\theta_2)\right\}\ldots\ldots\ldots\ldots(51).$$

71. From the formulæ given in the previous sections we can derive another case in which hyper-elliptic integrals of the first class depend on elliptic functions. For this purpose let us put $\sin\theta=i\tan\omega$ in the equations (43), (44); by this transformation $L(\kappa',\kappa'',\kappa'\kappa'',\theta)$ becomes

$$iL\{\sqrt{(1-\kappa'^2)},\sqrt{(1-\kappa''^2)},\sqrt{(1-\kappa'^2\kappa''^2)},\omega\}$$

$$-iM\{\sqrt{(1-\kappa'^2)},\sqrt{(1-\kappa''^2)},\sqrt{(1-\kappa'^2\kappa''^2)},\omega\},$$

and $M(\kappa',\kappa'',\kappa'\kappa'',\theta)$ becomes

$$-iM\{\sqrt{(1-\kappa'^2)},\sqrt{(1-\kappa''^2)},\sqrt{(1-\kappa'^2\kappa''^2)},\omega\};$$

and if we put

$$\tan\omega_1=\frac{(1+\kappa'\kappa'')\tan\omega}{1-\kappa'\kappa''\tan^2\omega},\quad \tan\omega_2=\frac{(1-\kappa'\kappa'')\tan\omega}{1+\kappa'\kappa''\tan^2\omega};$$

the same transformation gives $\sin\theta_1=i\tan\omega_1$, $\sin\theta_2=i\tan\omega_2$. Now by the employment of these latter equations

$$F(m_1,\theta_1)\text{ becomes }iF\{\sqrt{(1-m_1^2)},\omega_1\},$$

and

$$F(m_2,\theta_2)\text{ becomes }iF\{\sqrt{(1-m_2^2)},\omega_2\};$$

so that the equations (43), (44) when transformed by the equation $\sin\theta=i\tan\omega$ lead to

$$L\{\sqrt{(1-\kappa'^2)},\sqrt{(1-\kappa''^2)},\sqrt{(1-\kappa'^2\kappa''^2)},\omega\}$$

$$=\frac{1}{2\kappa'\kappa''}\left[F\left\{\frac{\sqrt{\{(1-\kappa'^2)(1-\kappa''^2)\}}}{1+\kappa'\kappa''},\omega_1\right\}\right.$$

$$\left.-F\left\{\frac{\sqrt{\{(1-\kappa'^2)(1-\kappa''^2)\}}}{1-\kappa'\kappa''},\omega_2\right\}\right]\ldots\ldots(52),$$

$$M\{\sqrt{(1-\kappa'^2)},\ \sqrt{(1-\kappa''^2)},\ \sqrt{(1-\kappa'^2\kappa''^2)},\ \omega\}$$

$$=\frac{1}{2\kappa'\kappa''}\left[\frac{1}{1+\kappa'\kappa''}F\left\{\frac{\sqrt{\{(1-\kappa'^2)(1-\kappa''^2)\}}}{1+\kappa'\kappa''},\ \omega_1\right\}\right.$$

$$\left.-\frac{1}{1-\kappa'\kappa''}F\left\{\frac{\sqrt{\{(1-\kappa'^2)(1-\kappa''^2)\}}}{1-\kappa'\kappa''},\ \omega_2\right\}\right]\text{......(53).}$$

72. To express these results in a manner conformable to our ordinary notation, according to which we write the hyperelliptic functions as $L(\kappa, \kappa', \kappa'', \omega)$, $M(\kappa, \kappa', \kappa'', \omega)$ where $\kappa'' > \kappa' > \kappa$, we shall remark that $1-\kappa'^2\kappa''^2 > 1-\kappa'^2 > 1-\kappa''^2$, and if we put for a moment $1-\kappa'^2\kappa''^2=\xi''^2$, $1-\kappa'^2=\xi'^2$, $1-\kappa''^2=\xi^2$, we find $\xi''^2=\xi^2+\xi'^2-\xi^2\xi'^2$, and the above formulæ become

$$L\{\xi,\ \xi',\ \sqrt{(\xi^2+\xi'^2-\xi^2\xi'^2)},\ \omega\}$$

$$=\frac{1}{2\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}}\left[F\left\{\frac{\xi\xi'}{1+\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}},\ \omega_1\right\}\right.$$

$$\left.-F\left\{\frac{\xi\xi'}{1-\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}},\ \omega_2\right\}\right],$$

$$M\{\xi,\ \xi',\ \sqrt{(\xi^2+\xi'^2-\xi^2\xi'^2)},\ \omega\}$$

$$=\frac{1}{2\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}}$$

$$\times\left[\frac{1}{1+\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}}F\left\{\frac{\xi\xi'}{1+\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}},\ \omega_1\right\}\right.$$

$$\left.-\frac{1}{1-\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}}F\left\{\frac{\xi\xi'}{1-\sqrt{\{(1-\xi^2)(1-\xi'^2)\}}},\ \omega_2\right\}\right],$$

in which, if we write κ, κ' in place of ξ, ξ', and k, k' in place of $\sqrt{(1-\xi^2)}$, $\sqrt{(1-\xi'^2)}$, we have the following, which are expressed in our ordinary notation:

$$L\{\kappa,\ \kappa',\ \sqrt{(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)},\ \omega\}$$

$$=\frac{1}{2kk'}\left\{F\left(\frac{\kappa\kappa'}{1+kk'},\ \omega_1\right)-F\left(\frac{\kappa\kappa'}{1-kk'},\ \omega_2\right)\right\}\text{...(54),}$$

$$M\{\kappa,\ \kappa',\ \sqrt{(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)},\ \omega\}$$

$$=\frac{1}{2kk'}\left\{\frac{1}{1+kk'}F\left(\frac{\kappa\kappa'}{1+kk'};\ \omega_1\right)-\frac{1}{1-kk'}F\left(\frac{\kappa\kappa'}{1-kk'},\ \omega_2\right)\right\}$$

$$\dots\dots\dots\dots\dots\dots\dots(55),$$

and $\tan\omega_1=\dfrac{(1+kk')\tan\omega}{1-kk'\tan^2\omega}$, $\tan\omega_2=\dfrac{(1-kk')\tan\omega}{1+kk'\tan^2\omega}$...(56).

Hence hyper-elliptic functions of the form

$\alpha L\{\kappa,\ \kappa',\ \sqrt{(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)},\ \omega\}+\beta M\{\kappa,\ \kappa',\ \sqrt{(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)},\ \omega\}$ can be added by the formula for the addition of elliptic functions of the form $F\left(\dfrac{\kappa\kappa'}{1+kk'},\ \omega_1\right)$, and by the formula for the addition of functions of the form $F\left(\dfrac{\kappa\kappa'}{1-kk'},\ \omega_2\right)$.

73. We now proceed to apply the transformation $\sin\theta=i\tan\omega$ to the equation (51). In the first place we remark that by this transformation

$$\int_0^\theta\frac{d\theta}{\cos^2\theta\sqrt{\{(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)(1-\kappa'^2\kappa''^2\sin^2\theta)\}}}$$

becomes

$$i\int_0^\omega\frac{\cos^4\omega d\omega}{\sqrt{[\{1-(1-\kappa'^2)\sin^2\omega\}\{1-(1-\kappa''^2)\sin^2\omega\}\{1-(1-\kappa'^2\kappa''^2)\sin^2\omega\}]}},$$

which is equal to

$$i\int_0^\omega\frac{(1-2\sin^2\omega+\sin^4\omega)\,d\omega}{\sqrt{[\{1-(1-\kappa'^2)\sin^2\omega\}\{1-(1-\kappa''^2)\sin^2\omega\}\{1-(1-\kappa'^2\kappa''^2)\sin^2\omega\}]}}.$$

To deduce the effect of this transformation on the right-hand side of equation (51), we observe that

$\tan\theta_1\sqrt{(1-m_1^2\sin^2\theta_1)}$ becomes $i\tan\omega_1\sqrt{\{1-(1-m_1^2)\sin^2\omega_1\}}$,

$E(m_1,\theta_1)$ becomes

$$i\,[\tan\omega_1\sqrt{\{1-(1-m_1^2)\sin^2\omega_1\}}$$
$$+F\{\sqrt{(1-m_1^2)},\ \omega_1\}-E\{\sqrt{(1-m_1^2)},\ \omega_1\}]$$

(28), and $F(m_1,\ \theta_1)$ becomes $iF\{\sqrt{(1-m_1^2)},\ \omega_1\}$; similar relations are also deduced by replacing m_1, θ_1, ω_1 by m_2, θ_2, ω_2;

the right-hand side of equation (51) is thus seen to be transformed into

$$\frac{i}{2(1-\kappa'^2)(1-\kappa''^2)}\left[E(\sqrt{(1-m_1^2)},\omega_1)-F(\sqrt{(1-m_1^2)},\omega_1)\right.$$
$$\left.+E(\sqrt{(1-m_2^2)},\omega_2)-F(\sqrt{(1-m_2^2)},\omega_2)\right]$$
$$+\frac{i}{2}\left[\frac{1}{1+\kappa'\kappa''}F(\sqrt{(1-m_1^2)},\omega_1)+\frac{1}{1-\kappa'\kappa''}F(\sqrt{(1-m_2^2)},\omega_2)\right],$$

which is equal to

$$i\int_0^\omega \frac{(1-2\sin^2\omega+\sin^4\omega)\,d\omega}{\sqrt{\{1-(1-\kappa'^2)\sin^2\omega\}\{1-(1-\kappa''^2)\sin^2\omega\}\{1-(1-\kappa'^2\kappa''^2)\sin^2\omega\}}},$$

and by taking into account the expressions for

$$\int_0^\omega \frac{d\omega}{\sqrt{\{1-(1-\kappa'^2)\sin^2\omega\}\{1-(1-\kappa''^2)\sin^2\omega\}\{1-(1-\kappa'^2\kappa''^2)\sin^2\omega\}}},$$

$$\int_0^\omega \frac{\sin^2\omega d\omega}{\sqrt{\{1-(1-\kappa'^2)\sin^2\omega\}\{1-(1-\kappa''^2)\sin^2\omega\}\{1-(1-\kappa'^2\kappa''^2)\sin^2\omega\}}},$$

already given (52), (53) we find

$$\int_0^\omega \frac{\sin^4\omega d\omega}{\sqrt{(1-(1-\kappa'^2)\sin^2\omega)(1-(1-\kappa''^2)\sin^2\omega)(1-(1-\kappa'^2\kappa''^2)\sin^2\omega)}}$$
$$=\frac{1}{2(1-\kappa'^2)(1-\kappa''^2)}\left[E(\sqrt{(1-m_1^2)},\omega_1)+E(\sqrt{(1-m_2^2)},\omega_2)\right]$$
$$+\frac{1}{2\kappa'\kappa''(1-\kappa'^2)(1-\kappa''^2)}\left[\frac{1-\kappa'^2-\kappa''^2-\kappa'\kappa''}{1+\kappa'\kappa''}F(\sqrt{(1-m_1^2)},\omega_1)\right.$$
$$\left.-\frac{1-\kappa'^2-\kappa''^2+\kappa'\kappa''}{1-\kappa'\kappa''}F(\sqrt{(1-m_2^2)},\omega_2)\right].$$

74. In this equation we shall put, as before, $1-\kappa'^2\kappa''^2=\xi''^2$, $1-\kappa'^2=\xi'^2$, $1-\kappa''^2=\xi^2$, and it becomes

$$\int_0^\omega \frac{\sin^4\omega d\omega}{\sqrt{(1-\xi^2\sin^2\omega)(1-\xi'^2\sin^2\omega)(1-(\xi^2+\xi'^2-\xi^2\xi'^2)\sin^2\omega)}}$$
$$=\frac{1}{2\xi^2\xi'^2}\left[E\left(\frac{\xi\xi'}{1+\sqrt{(1-\xi^2)(1-\xi'^2)}},\omega_1\right)\right.$$
$$\left.+E\left(\frac{\xi\xi'}{1-\sqrt{(1-\xi^2)(1-\xi'^2)}},\omega_2\right)\right]$$

$$+\frac{1}{2\xi^2\xi'^2\sqrt{(1-\xi^2)(1-\xi'^2)}}$$

$$\times\left[\frac{\xi^2+\xi'^2-1-\sqrt{(1-\xi^2)(1-\xi'^2)}}{1+\sqrt{(1-\xi^2)(1-\xi'^2)}}F\left\{\frac{\xi\xi'}{1+\sqrt{(1-\xi^2)(1-\xi'^2)}},\,\omega_1\right\}\right.$$

$$\left.-\frac{\xi^2+\xi'^2-1+\sqrt{(1-\xi^2)(1-\xi'^2)}}{1-\sqrt{(1-\xi^2)(1-\xi'^2)}}F\left\{\frac{\xi\xi'}{1-\sqrt{(1-\xi^2)(1-\xi'^2)}},\,\omega_2\right\}\right].$$

This equation can be put in conformity with our usual notation by writing κ and κ' respectively for ξ and ξ', and by writing k and k' for $\sqrt{(1-\xi^2)}$, $\sqrt{(1-\xi'^2)}$. We thus have

$$\int_0^\omega \frac{\sin^4\omega d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$

$$=\frac{1}{2\kappa^2\kappa'^2}\left\{E\left(\frac{\kappa\kappa'}{1+kk'},\,\omega_1\right)+E\left(\frac{\kappa\kappa'}{1-kk'},\,\omega_2\right)\right\}$$

$$+\frac{1}{2\kappa^2\kappa'^2kk'}\left\{\frac{\kappa^2+\kappa'^2-1-kk'}{1+kk'}F\left(\frac{\kappa\kappa'}{1+kk'},\,\omega_1\right)\right.$$

$$\left.-\frac{\kappa^2+\kappa'^2-1+kk'}{1-kk'}F\left(\frac{\kappa\kappa'}{1-kk'},\,\omega_2\right)\right\}\text{......(57).}$$

75. To find the value of the definite integral

$$\int_0^{\frac{1}{2}\pi}\frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$

we must remark that if we put $\tan\lambda=\frac{1}{\sqrt{(kk')}}$, it follows from (56) that ω_1 continues positive from $\omega=0$ to $\omega=\lambda$, and is negative from $\omega=\lambda$ to $\omega=\frac{1}{2}\pi$. Let a be the value of ω_2 corresponding to $\omega=\lambda$. Now we have

$$\int_0^{\frac{1}{2}\pi}\frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$

$$=\int_0^{\lambda}\frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$

$$+\int_\lambda^{\frac{1}{2}\pi}\frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}},$$

and if we put $\eta_1 = \frac{\kappa\kappa'}{1+kk'}$, $\eta_2 = \frac{\kappa\kappa'}{1-kk'}$ we deduce from (54)

$$\int_0^{\lambda} \frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$
$$= \frac{1}{2kk'}\left\{\int_0^{\frac{1}{2}\pi} \frac{d\omega_1}{\sqrt{(1-\eta_1^2\sin^2\omega_1)}} - \int_0^{\alpha} \frac{d\omega_2}{\sqrt{(1-\eta_2^2\sin^2\omega_2)}}\right\},$$

$$\int_{\lambda}^{\frac{1}{2}\pi} \frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$
$$= \frac{1}{2kk'}\left\{\int_{\frac{1}{2}\pi}^{0} \frac{d\omega_1}{\sqrt{(1-\eta_1^2\sin^2\omega_1)}} - \int_{\alpha}^{0} \frac{d\omega_2}{\sqrt{(1-\eta_2^2\sin^2\omega_2)}}\right\};$$

we find consequently

$$\int_0^{\frac{1}{2}\pi} \frac{d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$
$$= \frac{1}{kk'}\int_0^{\frac{1}{2}\pi} \frac{d\omega_1}{\sqrt{(1-\eta_1^2\sin^2\omega_1)}}.$$

In the same way we deduce from (55)

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\omega\, d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$
$$= \frac{1}{kk'(1+kk')}\int_0^{\frac{1}{2}\pi} \frac{d\omega_1}{\sqrt{(1-\eta_1^2\sin^2\omega_1)}},$$

and from (57)

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\omega\, d\omega}{\sqrt{(1-\kappa^2\sin^2\omega)(1-\kappa'^2\sin^2\omega)(1-(\kappa^2+\kappa'^2-\kappa^2\kappa'^2)\sin^2\omega)}}$$
$$= \frac{1}{\kappa^2\kappa'^2}\left\{\int_0^{\frac{1}{2}\pi} \sqrt{(1-\eta_1^2\sin^2\omega_1)}\, d\omega_1 + \frac{\kappa^2+\kappa'^2-1-kk'}{kk'(1+kk')}\int_0^{\frac{1}{2}\pi} \frac{d\omega_1}{\sqrt{(1-\eta_1^2\sin^2\omega_1)}}\right\}.$$

76. We have already shewn (21) that $L(\kappa, \kappa', \kappa'', \theta)$ and $M(\kappa, \kappa', \kappa'', \theta)$ depend on

$$L\left\{\sqrt{\left(\frac{\kappa''^2-\kappa'^2}{1-\kappa'^2}\right)}, \sqrt{\left(\frac{\kappa''^2-\kappa^2}{1-\kappa^2}\right)}, \kappa'', \phi\right\},$$

and on $$M\left\{\sqrt{\left(\frac{\kappa''^2-\kappa'^2}{1-\kappa'^2}\right)}, \sqrt{\left(\frac{\kappa''^2-\kappa^2}{1-\kappa^2}\right)}, \kappa'', \phi\right\},$$

if $k'' \tan\theta \tan\phi = 1$, and since

$$\kappa'' > \sqrt{\left(\frac{\kappa''^2 - \kappa^2}{1 - \kappa^2}\right)} > \sqrt{\left(\frac{\kappa''^2 - \kappa'^2}{1 - \kappa'^2}\right)},$$

we see that the hyper-elliptic functions $L(\kappa, \kappa', \kappa'', \theta)$ and $M(\kappa, \kappa', \kappa'', \theta)$ depend on elliptic functions if

$$\frac{\kappa''^2 (\kappa''^2 - \kappa^2)}{1 - \kappa^2} = \frac{\kappa''^2 - \kappa'^2}{1 - \kappa'^2},$$

and this latter equation gives

$$\kappa''^2 = \frac{\kappa'^2 (1 - \kappa^2)}{1 - \kappa'^2}, \text{ or } \kappa'' = \frac{k\kappa'}{k'} \text{ .... (58).}$$

CHAPTER VII.

APPLICATION OF THE PRECEDING RESULTS TO THE RECTIFICATION OF THE LINES OF CURVATURE OF AN ELLIPSOID.

77. WE suppose the following equations in rectangular coordinates

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-b^2}-\frac{z^2}{a^2-c^2}=1,$$

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}-\frac{z^2}{c^2-\mu^2}=1,$$

$$\frac{x^2}{\nu^2}-\frac{y^2}{b^2-\nu^2}-\frac{z^2}{c^2-\nu^2}=1,$$

to represent a system of confocal surfaces of the second order or quadrics, as they have been recently designated. The first of these equations belongs to the ellipsoid of the system, whose semi-axes in order of decreasing magnitude are a, $\sqrt{(a^2-b^2)}$, $\sqrt{(a^2-c^2)}$, so that $c>b$; the second of these equations represents the hyperboloid of one sheet, which, as it is a ruled surface, is sometimes called the *gauche* hyperboloid; and the third is the equation of the hyperboloid of two sheets of the confocal system. If we solve for x^2, y^2, z^2 from the above system we deduce

$$x^2=\frac{a^2\mu^2\nu^2}{b^2c^2},\ y^2=\frac{(a^2-b^2)(\mu^2-b^2)(b^2-\nu^2)}{b^2(c^2-b^2)},\ z^2=\frac{(a^2-c^2)(c^2-\mu^2)(c^2-\nu^2)}{c^2(c^2-b^2)},$$

and if we establish any relation between μ and ν, these equations are the equations of a curve traced on the surface of the ellipsoid. Let us suppose μ to be constant; the equations just written then belong to the intersection of the ellipsoid with a confocal gauche hyperboloid; they represent one of its lines of curvature, any point of which is determined by the value of ν, and the expression for the element of its arc is

$$\sqrt{(dx^2+dy^2+dz^2)}=\sqrt{\left(\frac{(a^2-\nu^2)(\mu^2-\nu^2)}{(b^2-\nu^2)(c^2-\nu^2)}\right)}\,d\nu\ldots(59).$$

In the same way we find for the element of the arc of a line of curvature situated on an hyperboloid of two sheets the expression

$$\sqrt{(dx^2+dy^2+dz^2)}=\sqrt{\left(\frac{(a^2-\mu^2)(\mu^2-\nu^2)}{(\mu^2-b^2)(c^2-\mu^2)}\right)}\,d\mu\ldots(60),$$

and in both cases the arcs can be expressed by means of the hyper-elliptic integrals which we have been considering.

78. To determine the expression for the arc of a line of curvature situated on a gauche hyperboloid (μ constant), we shall put $\nu=b\sin\theta$ in (59), and the limits of ν being zero and b, the corresponding limits of θ are zero and $\frac{1}{2}\pi$. If we denote by s_θ the arc of the line of curvature which we are considering, we find then

$$ds_\theta=\sqrt{\left\{\frac{(a^2-b^2\sin^2\theta)(\mu^2-b^2\sin^2\theta)}{(c^2-b^2\sin^2\theta)}\right\}}\,d\theta;$$

if we put $\kappa=\frac{b}{a}$, $\kappa'=\frac{b}{c}$, $\kappa''=\frac{b}{\mu}$, we have $\kappa''>\kappa'>\kappa$, since μ is less than c and greater than b; and we find

$$ds_\theta=b\left\{\frac{\kappa'}{\kappa\kappa''}-\kappa'\left(\frac{\kappa''}{\kappa}+\frac{\kappa}{\kappa''}\right)\sin^2\theta+\kappa\kappa'\kappa''\sin^4\theta\right\}$$
$$\times\frac{d\theta}{\sqrt{(1-\kappa^2\sin^2\theta)(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)}}\ldots(61),$$

so that the arc of the line of curvature counted from the vertex where it meets the plane of the mean and least semi-axes of the ellipsoid, is

$$b\left\{\frac{\kappa'}{\kappa\kappa''}L(\kappa,\kappa',\kappa'',\theta)-\kappa'\left(\frac{\kappa''}{\kappa}+\frac{\kappa}{\kappa''}\right)M(\kappa,\kappa',\kappa'',\theta)\right.$$
$$\left.+\kappa\kappa'\kappa''N(\kappa,\kappa',\kappa'',\theta)\right\}.$$

If therefore s_ϕ, s_ψ, s_χ, s_σ, s_τ denote five such arcs whose variable extremities are connected by the equations

$$\cos\sigma\,\cos\tau = \cos\phi\,\cos\psi\,\cos\chi - \sin\phi\,\sin\psi\,\sin\chi\nabla(\sigma,\tau),$$
$$\cos\phi\,\cos\psi = \cos\sigma\,\cos\tau\,\cos\chi + \sin\sigma\,\sin\tau\,\sin\chi\nabla(\phi,\psi),$$

we have in consequence of what has been proved in sections (17), (18)

$$s_\phi + s_\psi + s_\chi - s_\sigma - s_\tau = b\kappa\kappa'\kappa''\sin\phi\,\sin\psi\,\sin\chi\,\sin\sigma\,\sin\tau,$$

and if x', x'', x''', x^{iv}, x^{v} are the x's of the variable extremities of these arcs, we have

$$s_\phi + s_\psi + s_\chi - s_\sigma - s_\tau = \frac{b^4c^4}{a^6\mu^6}x'x''x'''x^{\text{iv}}x^{\text{v}}.$$

79. We can obtain another expression for the arc of a line of curvature which lies on a gauche hyperboloid (μ) by transforming the expression for the differential of its arc (61) by the equation $k''\tan\theta\tan\theta' = 1$, and by availing ourselves of (21), (23).

Since $\theta' = 0$ corresponds to the vertex of the curve situated in the plane of the greatest and least semi-axes of the ellipsoid, if the arc $(s_{\theta'})$ is counted from this point, we have

$$ds_{\theta'} = \frac{b}{kk'}\left\{\begin{array}{l} \dfrac{\kappa'k''^2(\kappa''^2-\kappa^2)}{\kappa\kappa''^3}\cdot\dfrac{d\theta'}{\sqrt{\left(1-\dfrac{\kappa''^2-\kappa^2}{k^2}\sin^2\theta'\right)\left(1-\dfrac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\theta'\right)\left(1-\kappa''^2\sin^2\theta'\right)}} \\ +\dfrac{\kappa\kappa'k''^4}{\kappa''^3}\cdot\dfrac{d\theta'}{(1-\kappa''^2\sin^2\theta')\sqrt{\left(1-\dfrac{\kappa''^2-\kappa^2}{k^2}\sin^2\theta'\right)\left(1-\dfrac{\kappa''^2-\kappa'^2}{k'^2}\sin^2\theta'\right)\left(1-\kappa''^2\sin^2\theta'\right)}} \end{array}\right\},$$

and by integrating and making use of our previous notation, as defined in Chapter III., we have

$$s_{\theta'}=\frac{b}{kk'}\left[\frac{\kappa' k''^2(\kappa''^2-\kappa^2)}{\kappa\kappa''^3}\,L\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\theta'\right)+\frac{\kappa\kappa' k''^4}{\kappa''^3}\,P\left(-\kappa''^2,\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k'},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\theta'\right)\right].$$

If then $\phi', \psi', \chi', \sigma', \tau'$ satisfy the transcendental system

$$L\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\phi'\right)+L\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\psi'\right)+L\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\chi'\right)$$

$$-L\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\ \kappa'',\ \sigma'\right)-L\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\ \kappa'',\ \tau'\right)=0,$$

$$M\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\phi'\right)+M\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\psi'\right)+M\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\kappa'',\chi'\right)$$

$$-M\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\ \kappa'',\ \sigma'\right)-M\left(\frac{\sqrt{(\kappa''^2-\kappa^2)}}{k},\ \frac{\sqrt{(\kappa''^2-\kappa'^2)}}{k'},\ \kappa'',\ \tau'\right)=0,$$

or any of its algebraic equivalents, we find, in consequence of the formula given for the addition of the function P (18), if $s_{\phi'}$, $s_{\psi'}$, &c. denote the arcs corresponding to the amplitudes ϕ', ψ', &c.

$$s_{\phi'}+s_{\psi'}+s_{\chi'}-s_{\sigma'}-s_{\tau'}=\frac{b\kappa\kappa'\kappa'' k''^4}{kk'}\ \frac{\sin\phi'\ \sin\psi'\ \sin\chi'\ \sin\sigma'\ \sin\tau'}{\sqrt{(1-\kappa''^2\sin^2\phi')(1-\kappa''^2\sin^2\psi')(1-\kappa''^2\sin^2\chi')(1-\kappa''^2\sin^2\sigma')(1-\kappa''^2\sin^2\tau')}},$$

or if ϕ', ψ', &c. denote angles which are derived from ϕ, ψ, &c., in the same way as θ' is derived from θ, we have

$$s_{\phi'} + s_{\psi'} + s_{\chi'} - s_{\sigma'} - s_{\tau'} = \frac{b\kappa\kappa'\kappa''}{kk'k''} \cos\phi \cos\psi \cos\chi \cos\sigma \cos\tau.$$

If y', y'', y''', y^{iv}, y^{v} denote the y's of the variable extremities of these arcs, we deduce from this last expression

$$s_{\phi'} + s_{\psi'} + s_{\chi'} - s_{\sigma'} - s_{\tau'} = \frac{b^4(c^2-b^2)^2}{\{(a^2-b^2)(\mu^2-b^2)\}^3} y'y''y'''y^{iv}y^{v}.$$

80. To determine the particular μ lines of curvature, whose rectification depends on elliptic functions, we shall avail ourselves of the formulæ given in the last chapter. By referring to the values of κ, κ', κ'' in terms of the constants of the ellipsoid given in section (78), we see that the equation $\kappa = \kappa'\kappa''$ gives $\mu = \frac{ab}{c}$; and as $c > \mu$, this line of curvature does not exist on the ellipsoid unless $c^2 - ab > 0$. Again, in consequence of (54), (55), (57), if $\kappa'''^2 = \kappa^2 + \kappa'^2 - \kappa^2\kappa'^2$ the value of μ resulting from this relation will determine a line of curvature, whose rectification depends on elliptic functions. This value is easily seen to be $\frac{ac}{\sqrt{(a^2+c^2-b^2)}}$, and as this quantity is intermediate between c and b, such a line of curvature always exists and is situated on the gauche hyperboloid whose equation is

$$\frac{x^2}{a^2c^2} + \frac{y^2}{(a^2-b^2)(c^2-b^2)} - \frac{z^2}{c^2(c^2-b^2)} = \frac{1}{a^2+c^2-b^2}.$$

Again, in consequence of (58) the value of μ derived from the equation $\kappa'''^2 = \frac{\kappa'^2(1-\kappa^2)}{1-\kappa'^2}$ will determine a line of curvature, whose rectification depends on elliptic functions: this value is $a\sqrt{\left(\frac{c^2-b^2}{a^2-b^2}\right)}$, and will not render $\mu^2 - b^2 > 0$ unless $a^2 - b^2 - a\sqrt{(a^2-c^2)} > 0$, or unless the mean semi-axis is greater than the mean proportional between the greatest and least semi-axes, in which case such a line of curvature exists.

81. If in the expression (60) for the differential of the arc of a line of curvature of the ellipsoid, which is situated on a confocal hyperboloid of two sheets (ν), we put

$$\eta^2 = \frac{c^2 - b^2}{a^2 - b^2}, \quad \eta'^2 = \frac{a^2\eta^2}{c^2}, \quad \eta''^2 = \frac{(a^2 - \nu^2)\,\eta^2}{c^2 - \nu^2}, \quad \mu^2 = \frac{c^2 - a^2\eta^2 \sin^2\theta}{1 - \eta^2 \sin^2\theta},$$

we have $$ds = \frac{a^2 - c^2}{c\sqrt{(a^2 - b^2)\,(c^2 - \nu)^2}} \left\{\frac{a^2 - c^2}{1 - \eta^2 \sin^2\theta} - (a^2 - \nu^2)\right\}$$
$$\times \frac{d\theta}{\sqrt{(1 - \eta^2 \sin^2\theta)\,(1 - \eta'^2 \sin^2\theta)\,(1 - \eta''^2 \sin^2\theta)}};$$

η, η', η'' are less than unity and $\eta'' > \eta' > \eta$.

Since $\theta = 0$ corresponds to the vertex of the curve which is situated in the plane of the greatest and mean semi-axes, if s_θ denotes the arc counted from this point, we have

$$s_\theta = \frac{a^2 - c^2}{c\sqrt{(a^2 - b^2)\,(c^2 - \nu^2)}}$$
$$\times \{(a^2 - c^2)\,P(-\eta^2, \eta, \eta', \eta'', \theta) - (a^2 - \nu^2)\,L(\eta, \eta', \eta'', \theta)\} \ldots (62).$$

82. If ϕ, ψ, χ, σ, τ denote five angles satisfying the transcendental system

$$L(\eta, \eta', \eta'', \phi) + L(\eta, \eta', \eta'', \psi) + L(\eta, \eta', \eta'', \chi)$$
$$- L(\eta, \eta', \eta'', \sigma) - L(\eta, \eta', \eta'', \tau) = 0,$$
$$M(\eta, \eta', \eta'', \phi) + M(\eta, \eta', \eta'', \psi) + M(\eta, \eta', \eta'', \chi)$$
$$- M(\eta, \eta', \eta'', \sigma) - M(\eta, \eta', \eta'', \tau) = 0,$$

or any of the algebraic systems which are equivalent to it, and if s_ϕ, s_ψ, &c. denote the corresponding arcs of a line of curvature situated on the surface (ν), we find in consequence of the formula (18) given for the addition of the function P

$$s_\phi + s_\psi + s_\chi - s_\sigma - s_\tau = \frac{(a^2 - c^2)^2}{c\sqrt{\{(a^2 - b^2)\,(c^2 - \nu^2)\}}}$$
$$\times \frac{\eta^4 \sin\phi \sin\psi \sin\chi \sin\sigma \sin\tau}{\sqrt{(1-\eta^2\sin^2\phi)(1-\eta^2\sin^2\psi)(1-\eta^2\sin^2\chi)(1-\eta^2\sin^2\sigma)(1-\eta^2\sin^2\tau)}}.$$

83. We have the following equation connecting ϕ with the corresponding value of μ, which we shall call μ_1,

$$\frac{\sin\phi}{\sqrt{(1-\eta^2\sin^2\phi)}}=\sqrt{\left\{\frac{(a^2-b^2)(c^2-\mu_1^2)}{(a^2-c^2)(c^2-b^2)}\right\}},$$

whence if z' is the z of this point, which is the variable extremity of the arc s_ϕ, we deduce

$$\frac{\sin\phi}{\sqrt{(1-\eta^2\sin^2\phi)}}=\frac{c\sqrt{(a^2-b^2)}}{(a^2-c^2)\sqrt{(c^2-\nu^2)}}z'.$$

Let z'', z''', z^{iv}, z^{v} be the z's of the variable extremities of the arcs s_ψ, s_χ, &c.; in consequence of what has been just proved, we have

$$\frac{\sin\psi}{\sqrt{(1-\eta^2\sin^2\psi)}}=\frac{c\sqrt{(a^2-b^2)}}{(a^2-c^2)\sqrt{(c^2-\nu^2)}}z'',$$

$$\frac{\sin\chi}{\sqrt{(1-\eta^2\sin^2\chi)}}=\frac{c\sqrt{(a^2-b^2)}}{(a^2-c^2)\sqrt{(c^2-\nu^2)}}z''',$$

and so on, so that finally we find

$$s_\phi+s_\psi+s_\chi-s_\sigma-s_\tau=\frac{c^4(c^2-b^2)^2}{\{(a^2-c^2)(c^2-\nu^2)\}^3}z'z''z'''z^{iv}z^{v}.$$

84. By the formula (20), given in Chapter III., $P(-\eta^2,\eta,\eta',\eta'',\theta)$ can be expressed by the functions $L(\eta,\eta',\eta'',\theta)$, $M(\eta,\eta',\eta'',\theta)$, $N(\eta,\eta',\eta'',\theta)$, $Q(\eta,\eta',\eta'',\theta)$; so that the arc of the line of curvature which we are considering can be made to depend on these functions by substituting for $P(-\eta^2,\eta,\eta',\eta'',\theta)$ in (62) this expression. The formula thence resulting for the comparison of the arcs of a line of curvature of an ellipsoid, which is situated on an hyperboloid of two sheets, is however rather complicated.

85. To find the lines of curvature of an ellipsoid lying on a confocal hyperboloid of two sheets, which are rectifiable by elliptic functions, we shall proceed as before. If we seek the value of ν derived from the equation $\eta=\eta'\eta''$, we find $\nu=\frac{\sqrt{\{b^2(a^2+c^2)-a^2c^2\}}}{b}$; such a line of curvature is

therefore not possible unless $b\sqrt{(a^2+c^2)} > ac$. Again, the value of ν derived from the condition $\eta''^2 = \dfrac{\eta'^2(1-\eta^2)}{1-\eta'^2}$ gives $\nu = a\sqrt{\left(\dfrac{c^2-b^2}{a^2-b^2}\right)}$; such a line of curvature is not possible unless $b > a\sqrt{\left(\dfrac{c^2-b^2}{a^2-b^2}\right)}$ or unless $a^2(a^2-c^2)-(a^2-b^2)^2 > 0$; and will not exist unless the mean proportional between the greatest and least semi-axes exceeds the square of the mean semi-axis. The curve of intersection of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2-b^2} + \dfrac{z^2}{a^2-c^2} = 1$ with the surface

$$\frac{x^2}{a^2(c^2-b^2)} + \frac{y^2}{(a^2-b^2)^2 - a^2(a^2-c^2)} - \frac{z^2}{b^2(a^2-c^2)} = \frac{1}{a^2-b^2}$$

will be rectifiable by elliptic functions, and this latter surface will be an hyperboloid of one or of two sheets according as $(a^2-b^2)^2 - a^2(a^2-c^2)$ is greater or less than zero.

86. The value of ν derived from the equation $\eta''^2 = \eta^2 + \eta'^2 - \eta^2\eta'^2$ is $\dfrac{bc}{a}$. The rectification of the corresponding line of curvature offers results of sufficient interest to justify our entering into the question in some detail. In this case the length of the arc is best derived at once from the formula (60), which gives us the following equation for the element of the arc, $ds = \dfrac{1}{a}\sqrt{\left\{\dfrac{(a^2-\mu^2)(a^2\mu^2-b^2c^2)}{(\mu^2-b^2)(c^2-\mu^2)}\right\}}\,d\mu$, and if we put $\mu = \sqrt{(bc)}\,t$, $\dfrac{a^2}{bc} = \alpha$, $\dfrac{b}{c} = \beta$, we find

$$ds = \frac{b\sqrt{(bc)}}{a}\sqrt{\left\{\frac{1-\alpha\left(t^2+\dfrac{1}{t^2}\right)+\alpha^2}{1-\beta\left(t^2+\dfrac{1}{t^2}\right)+\beta^2}\right\}}\,dt.$$

If $t + \dfrac{1}{t} = u$, we have $2t = u \pm \sqrt{(u^2-4)}$. For points of the arc of the curve which are situated between the vertex (O) for which $\mu = b$ and the point (P) at which $\mu = \sqrt{(bc)}$, we must use the lower sign in the radical entering into the expression for t: and for arcs of the curve counted from this latter point up to the vertex (Q) for which $\mu = c$, the upper sign is to be employed. Consequently the length of the arc counted from the vertex (O) to any point (T) of the arc OP is expressed by the integral

$$\tfrac{1}{2}\sqrt{(bc)}\int_{\frac{c+b}{\sqrt{(bc)}}}^{u}\sqrt{\left\{\frac{\left(\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}\right)^2-u^2}{\left(\sqrt{\beta}+\frac{1}{\sqrt{\beta}}\right)^2-u^2}\right\}}\left\{du-\frac{udu}{\sqrt{(u^2-4)}}\right\};$$

if we put

$$v=\sqrt{(u^2-4)},\quad u=\left(\sqrt{\beta}+\frac{1}{\sqrt{\beta}}\right)\sin\vartheta_1,\quad v=\left(\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\right)\sin\vartheta_2,$$

$$n_1=\frac{a(c+b)}{a^2+bc},\quad n_2=\frac{a(c-b)}{a^2-bc},$$

and integrate, remembering that when $\mu = b$, ϑ_1 and ϑ_2 both become $\frac{1}{2}\pi$, we find

$$\begin{aligned}\text{arc } OT=\tfrac{1}{2}\Bigg[&\frac{a^2-bc}{a}\{E(n_2,\tfrac{1}{2}\pi)-E(n_2,\vartheta_2)\}\\&-\frac{a^2+bc}{a}\{E(n_1,\tfrac{1}{2}\pi)-E(n_1,\vartheta_1)\}\Bigg].\end{aligned}$$

To find the length of the arc OP, let $\sin\lambda=\dfrac{2\sqrt{(bc)}}{b+c}$, and for the point P we have $\vartheta_1 = \lambda$, $\vartheta_2 = 0$, so that

$$\text{arc } OP=\tfrac{1}{2}\left[\frac{a^2-bc}{a}E(n_2,\tfrac{1}{2}\pi)-\frac{a^2+bc}{a}\{E(n_1,\tfrac{1}{2}\pi)-E(n_1,\lambda)\}\right].$$

The length of the arc counted from P to any point T' between P and the vertex Q is given by the integral

$$\frac{\sqrt{(bc)}}{2}\int_{2}^{u}\sqrt{\left\{\frac{\left(\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}\right)^2-u^2}{\left(\sqrt{\beta}+\frac{1}{\sqrt{\beta}}\right)^2-u^2}\right\}}\left\{du+\frac{udu}{\sqrt{(u^2-4)}}\right\},$$

so that we have

$$\text{arc } PT'=\tfrac{1}{2}\left[\frac{a^2+bc}{a}\{E(n_1,\vartheta_1)-E(n_1,\lambda)\}+\frac{a^2-bc}{a}E(n_2,\vartheta_2)\right].$$

For the vertex Q we have $\vartheta_1 = \vartheta_2 = \frac{1}{2}\pi$, so that

$$\text{arc } PQ = \tfrac{1}{2}\left[\frac{a^2+bc}{a}\{E(n, \tfrac{1}{2}\pi) - E(n_1, \lambda)\} + \frac{a^2-bc}{a} E(n_2, \tfrac{1}{2}\pi)\right].$$

By taking the sum of the arcs OP and PQ, we find for the quadrant of the line of curvature for which $\nu = \frac{bc}{a}$ the expression $\frac{a^2-bc}{a} E(n_2, \frac{1}{2}\pi)$; so that the circumference of this line of curvature is equal to the circumference of a plane ellipse whose greater semi-axis is $\frac{a^2-bc}{a}$, and whose eccentricity is $\frac{a(c-b)}{a^2-bc}$.

We find the following expression for the arc QT':

$$\text{arc } QT' = \tfrac{1}{2}\left[\frac{a^2-bc}{a}\{E(n_2, \tfrac{1}{2}\pi) - E(n_2, \vartheta_2)\} \right.$$
$$\left. + \frac{a^2+bc}{a}\{E(n_1, \tfrac{1}{2}\pi) - E(n_1, \vartheta_1)\}\right].$$

It is easy to see that the angles ϑ_1 and ϑ_2 are connected by the relation $\frac{\cos\vartheta_1}{\cos\vartheta_2} = \frac{c-b}{c+b}$.

Let V, V' be two points situated respectively on the arcs OP, PQ, and let μ, μ' be the focal semi-axes of the *gauches* hyperboloids which determine them; we shall call *corresponding points* two points so related, that for them $\mu\mu' = bc$: as this relation can be written $\frac{\mu}{\sqrt{(bc)}} = \frac{\sqrt{(bc)}}{\mu'}$, we see that at *corresponding* points the angles ϑ_1 and ϑ_2 have the same values.

Supposing that V and V' are *corresponding* points, we have

$$\text{arc } QV' + \text{arc } OV = \frac{a^2-bc}{a}\{E(n_2, \tfrac{1}{2}\pi) - E(n_2, \vartheta_2)\},$$

$$\text{arc } QV' - \text{arc } OV = \frac{a^2-bc}{a}\{E(n_1, \tfrac{1}{2}\pi) - E(n_1, \vartheta_1)\}.$$

If we subtract from the quadrant OQ the sum of the arc OV and QV', we find for the arc VV', which is terminated by two *corresponding* points, the expression $\frac{a^2-bc}{a}E(n_2, \vartheta_2)$; which is therefore equal to the arc of a plane ellipse, whose greater semi-axis is $\frac{a^2-bc}{a}$, and whose eccentricity is $\frac{a(c-b)}{a^2-bc}$.

Along the line of curvature, which we are considering, $\left(\text{for which } \nu = \frac{bc}{a}\right)$ the x's of every point are equal to the focal semi-axes of the corresponding *gauches* hyperboloids; so that if x', x'' are the x's of two *corresponding* points, we have $x'x'' = bc$.

If we put

$$\mu^2 = b^2\cos^2\phi + c^2\sin^2\phi, \quad \mu'^2 = b^2\cos^2\psi + c^2\sin^2\psi,$$

for *corresponding* points, we have

$$\tan\phi\tan\psi = \frac{b}{c}.$$

CONCLUSION.

GENERAL REMARKS.

87. FROM the formulæ given in Chapter II., it is plain that given any number of elliptic functions of the first kind with a common modulus, a single function of the same kind can be found which is equal to the sum of the given functions, the trigonometric functions of whose amplitude are algebraically connected with the trigonometric functions of the amplitudes of the given functions. From the results given in Chapter III. it appears that given any number of functions of the form $\alpha L(\kappa, \kappa', \kappa'', \theta) + \beta M(\kappa, \kappa', \kappa'', \theta)$ which arise from assigning arbitrary values to θ exceeding two, the sum of such functions can be expressed by two functions of the same sort. In general if

$$U(\theta)=\int_0^\theta \frac{\{a_0+a_1\sin^2\theta+\ldots+a_n\sin^{2n}\theta\}\,d\theta}{\sqrt{\{(1-\kappa^2\sin^2\theta)(1-\kappa'^2\sin^2\theta)(1-\kappa''^2\sin^2\theta)\ldots(1-\kappa^{(2n)^2}\sin^2\theta)\}}},$$

where $\kappa, \kappa', \kappa'', \ldots \kappa^{(2n)}$ are all less than unity, and $a_0, a_1, \ldots a_n$ are arbitrary quantities, the sum of any number (exceeding $n+1$) of such functions, which are formed by giving arbitrary values to θ, can be expressed by $n+1$ such functions. If

$$U(\theta)+U(\theta')+U(\theta'')+\ldots+U(\theta^{(n+1)})=U(\alpha')+U(\alpha'')+\ldots+U(\alpha^{(n+1)}),$$

we have (denoting by R the radical)

$$\int_0^\theta \frac{\sin^{2n+2}\theta d\theta}{R}+\int_{\alpha'}^{\theta'} \frac{\sin^{2n+2}\theta d\theta}{R}+\ldots+\int_{\alpha^{(n+1)}}^{\theta^{(n+1)}} \frac{\sin^{2n+2}\theta d\theta}{R}$$

$$=(-1)^{n+1}\sin\theta\sin\theta'\sin\theta''\ldots\sin\theta^{(n+1)}\sin\alpha'\sin\alpha''\ldots\sin\alpha^{(n+1)},$$

and $$\int_0^\theta \frac{\sin^{2(n+m)}\theta d\theta}{R}+\int_{\alpha'}^{\theta'} \frac{\sin^{2(n+m)}\theta d\theta}{R}+\ldots+\int_{\alpha^{(n+1)}}^{\theta^{(n+1)}} \frac{\sin^{2(n+m)}\theta d\theta}{R}$$

is a rational and integral function of the right side of the last equation.

APPENDIX.

NOTE (A), p. 8.

Haedenkamp has deduced by the method of elliptic coordinates the form under which Jacobi has presented Abel's theorem, see *Crelle*, Vol. XXV., p. 178. If $f(x)$ is of even dimensions in x, Jacobi has, by a slight modification of his demonstration, deduced the algebraic integrals of the transcendental system in this case. Retaining the notation in the text, it follows that $\Sigma \frac{d}{dx_1} \frac{\frac{f(x_1)}{a_r - x_1}}{\{\phi'(x_1)\}^2}$ is the coefficient of $\frac{1}{x}$ in the development of $\frac{\frac{f(x)}{a_r - x}}{\{\phi(x)\}^2}$. Now if $f(x) = a_0 + a_1 x + \ldots + a_{2m} x^{2m}$; this coefficient is $-a_{2m}$, so that the calculations in the text being suitably modified, we find

$$\frac{d^2u}{dt^2} + \frac{du^2}{dt^2} = -\tfrac{1}{4}a_{2m}, \text{ or } \frac{d^2 \surd\{\phi(a_r)\}}{dt^2} = \tfrac{1}{4}a_{2m} \surd\{\phi(a_r)\};$$

whence, by integration, $\left[\frac{d \surd\{\phi(a_r)\}}{dt}\right]^2 = \tfrac{1}{4}a_{2m}\phi(a_r) + c$; and if we substitute in this equation for $\phi(a_r)$ and $\frac{d}{dt}\surd\{\phi(a_r)\}$ their values in terms of $x_1, x_2, \ldots x_m$, we arrive at an algebraic integral of the transcendental system, which is derived from the root a_r; $(m-2)$ of the remaining roots will furnish the remaining independent integrals.

In this case t is no longer an algebraic function of $x_1, x_2, \ldots x_m$, but becomes a circular or logarithmic function of these quantities, and

$$\frac{x_1^{m-1}dx_1}{\surd\{f(x_1)\}} + \frac{x_2^{m-1}dx_2}{\surd\{f(x_2)\}} + \ldots + \frac{x_m^{m-1}dx_m}{\surd\{f(x_m)\}}$$

is the differential of a circular or logarithmic function of $x_1, x_2, \ldots x_m$.

NOTE (B), p. 68.

Legendre, I believe, first observed that $\int \frac{P\,dx}{\surd(a + \beta x^2 + \gamma x^4 + \beta x^6 + a x^8)}$ when P is a rational function of x, depends on elliptic functions. Jacobi subsequently remarked that $\int \frac{\sin^{2m}\theta\, d\theta}{\surd(1 - \kappa^2 \sin^2\theta)(1 - \kappa'^2 \sin^2\theta)(1 - \kappa^2\kappa'^2 \sin^2\theta)}$ depends on the same functions. The remaining conditions given in the text that the first class of hyper-elliptic integrals should be expressed by elliptic functions are given by Königsberger, see *Crelle*, vol. LXVII., p. 77. In a recent number of M. Brioschi's *Annali di Mathematica*, Mr. Gordan has shewn that if X denotes a sextic function whose skew invariant vanishes, the hyper-elliptic integral $\int\frac{(\alpha+\beta x)\,dx}{\sqrt{(X)}}$ depends on elliptic integrals. This elegant theorem can be thus simply demonstrated. If we put $x=\frac{p+qy}{p'+q'y}$, p, q, p', q' can be determined so that when X is expressed in terms of y, the absolute term and the coefficient of y^6 disappear. The above integral then takes the form

$$\int\frac{(\alpha'+\beta'y)\,\delta y}{\sqrt{y\,(1+2\lambda y+\mu y^2)\,(1+2\lambda' y+\mu' y^2)}},$$

but from the given condition, we have $\begin{vmatrix} 0, & 1, & 0 \\ 1, & \lambda, & \mu \\ 1, & \lambda', & \mu' \end{vmatrix}=0$, (see Salmon's *Higher Algebra*, p. 275), whence $\mu=\mu'$. If we assume then $y\sqrt{\mu}=t$, this integral is of the form

$$\int\frac{(\alpha''+\beta''t)\,dt}{t^{\frac{3}{2}}\sqrt{\left\{\left(t+\frac{1}{t}\right)+g\right\}\left\{\left(t+\frac{1}{t}\right)+h\right\}}},$$

and if $t+\frac{1}{t}=u$, we have $\sqrt{t}+\frac{1}{\sqrt{t}}=\sqrt{(u+2)}$, $\sqrt{t}-\frac{1}{\sqrt{t}}=\sqrt{(u-2)}$; so that

$$\frac{dt}{\sqrt{t}}=\tfrac{1}{2}\left\{\frac{du}{\sqrt{(u+2)}}+\frac{du}{\sqrt{(u-2)}}\right\},\quad \frac{dt}{t^{\frac{3}{2}}}=\tfrac{1}{2}\left\{\frac{du}{\sqrt{(u-2)}}-\frac{du}{\sqrt{(u+2)}}\right\},$$

and the integral which we are discussing depends on the integrals

$$\int\frac{du}{\sqrt{(u+g)\,(u+h)\,(u+2)}},\quad \int\frac{du}{\sqrt{(u+g)\,(u+h)\,(u-2)}}.$$

THE END.

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